Received: *June 23, 2000*, Accepted: *November 22, 2000*

Mirror symmetry by O3-planes

Bo Feng and Amihay Hanany

Center for Theoretical Physics, Massachusetts Institute of Technology
Cambridge MA 02139
E-mail: fengb@ctpyellow.mit.edu, hanany@mit.edu

Abstract: We construct the three dimensional mirror theory of $\mathrm{SO}(2k)$ and $\mathrm{SO}(2k+1)$ gauge groups by using O3-planes. An essential ingredient in constructing the mirror is the splitting of a physical brane (NS-brane or D5-brane) on O3-planes. In particular, matching the dimensions of moduli spaces of mirror pair (for example, the $\mathrm{SO}(2k+1)$ and its mirror) there is a D3-brane creation or annihilation accompanying the splitting. This novel dynamical process gives a nontrivial prediction for strongly coupled field theories, which will be very interesting to check by Seiberg-Witten curves. Furthermore, applying the same idea, we revisit the mirror theory of $\mathrm{Sp}(k)$ gauge group and find new mirrors which differ from previously known results. Our new result for $\mathrm{Sp}(k)$ gives another example to a previously observed fact, which shows that different theories can be mirror to the same theory. We also discussed the phenomena such as *"hidden FI-parameters"* when the number of flavors and the rank of the gauge group satisfy certain relations, *"incomplete higgsing"* for the mirror of $\mathrm{SO}(2k+1)$ and the *"hidden global symmetry"*. After discussing the mirror for a single Sp or SO gauge group, we extend the study to a product of two gauge groups in two different models, namely the elliptic and the non-elliptic models.

Keywords: Duality in Gauge Field Theories, String Duality, Brane Dynamics in Gauge Theories.

JHEP11(2000)033

Contents

1. Introduction

In [1], Intriligator and Seiberg found a new duality, the so-called "mirror symmetry", between two different $N = 4$ gauge theories in three dimensions. There exists such a mirror duality in three dimensions due to several special properties. First, the $N = 4$ theory has a global R-symmetry SO(4) which can be rewritten as $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, i.e., as the direct product of two independent SU(2) factors. This is one crucial property for mirror symmetry because one action of the mirror duality is to simply interchange these two SU(2) factors.[1] Under the global R-symmetry, the vector multiplet is in the adjoint of $\mathrm{SU}(2)_L$ and is invariant under $\mathrm{SU}(2)_R$ while the hypermultiplet is in the adjoint of $\mathrm{SU}(2)_R$ and is invariant under $\mathrm{SU}(2)_L$ (notice that both multiplets have four scalars if we dualize the gauge field A_μ in three dimensions to a scalar). Furthermore, the mass parameter transforms as $(3, 1)$ of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and the FI-parameter as $(1, 3)$. So after mirror duality, the Coulomb branch and mass parameter of one theory change to the Higgs branch and FI-parameter of the other and vice versa. Such mapping has an immediate application: because the Higgs branch is not renormalized by quantum effects [4], we can get the exact result about the Coulomb branch of one theory which is corrected by quantum effects by studying the Higgs branch of the mirror theory which can be studied at the classical level. Because of this and other good applications of mirror duality (for details, see [1]), a lot of work [5]–[13] has been done in this topic to try and find new mirror pairs.

There are several ways to construct the mirror pairs. The first way is to use the arguments coming from field theory [1, 5]. This method gives a lot of details how fields and parameters map to each other under the mirror duality. However, this method requires a lot of results which are not easy to get in field theory, so it is hard

[1] When we discuss the mirror duality of $N = 2$ theory in three dimensions, we must enhance the explicit U(1) global R-symmetry to two U(1)'s, i.e., $\mathrm{U}(1) \times \mathrm{U}(1)$. Otherwise there is no good way to define the mirror theory. For details see [2, 3].

to use it to construct general mirror pairs. The second way is to use M-theory to construct the mirror pairs as done by Porrati and Zaffaroni in [8]. The third way is to use the geometric realization in [14]. The fourth way, which is also the most popular way in the construction of mirror pairs, is given in [7] by using brane setups. The brane setup has the good property of making many quantities in field theory more visible. For example, the R-symmetry $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ corresponds to rotations in planes X^{345} and X^{789}. The Coulomb branch and Higgs branch become the positions of D3-branes in NS-branes and D5-branes. The mass parameter and FI-parameter also have similar geometric correspondences. These geometric pictures give us some intuition to understand the problem better (for more applications of brane setups, see review [15]). The key observation in [7] is that the mirror duality is just the S-duality in string theory. Using the known property of S-dual transformation of various kinds of branes [7, 6, 9, 10] we can easily find the mirror pairs. In this paper we will follow the last method.

Because we will use the brane setup to find the mirror theory, let us talk more about the general idea [7] of the brane construction. Given a gauge theory with gauge group and some matter contents, first we try to find a proper brane setup which represents the gauge theory (usually it is the Coulomb branch given explicitly in the brane setup). After that, we move to the Higgs branch[2] of the theory by splitting the D3-branes between NS-branes and D5-branes. Then we make the S-duality transformation (mirror transformation) which changes the NS-brane to D5-brane, D5-brane to NS-brane and D3-brane to itself, while perform the electric-magnetic duality in the world volume theory of D3-branes. When the brane setup involves an orientifold or ON plane, we need to know the S-duality rule for them too. Finally, we read out the corresponding gauge theory given by the S-dual brane setup — it is the mirror theory which we want to find.

In applications, it is straight forward to use the above procedure to give the mirror theory of $\mathrm{U}(n)$ gauge theory with some flavors or the product of $\mathrm{U}(n)$'s with some bifundamentals because the brane setup of those theories involve only NS5-branes, D5-branes and D3-branes and we know how to deal with them. However, when we try to find the mirror for a gauge group $\mathrm{Sp}(k)$ or $\mathrm{SO}(n)$, we must use an orientifold plane in the brane setup. Now a problem arises because sometimes we do not know how to read out the gauge theory of the S-dual brane setup of these orientifolds. The orientifolds which are involved in the construction can be divided into two types: the orientifold three plane (O3-plane) and the orientifold five plane (O5-plane). Sen has given an answer about the gauge theory under the ON-projection, which is the S-dual of the $O5^-$ plane plus a physical D5-brane, in [16]. Using this result, we can get the mirror theory for $\mathrm{Sp}(k)$ [9, 10] by using

[2]Usually, we can break all gauge symmetries by Higgs mechanism. However, in some cases after higgsing there are still some massless gauge fields. We call the latter case *"incomplete higgsing"*.

the orientifold five plane in the initial brane setup. For SO(k), if we insist on using the orientifold five plane again in the brane setup, we must know what is the gauge theory under the ON^+ projection which is the S-dual of $O5^+$ plane. It is still an open problem to read it out.

In the above paragraph, we mention that there is a difficulty to use orientifold five-plane to construct the mirror theory of SO(n) gauge group. However, for constructing the Sp(k) or SO(n) gauge theory we can use an O3-plane instead of the O5-plane. Because under S-duality the O3-plane changes into another O3-plane, we know how to read out the gauge theory (unlike the O5-plane which becomes ON plane under S-duality). Motivated by this observation, in this paper we use O3-planes to investigate the mirror theory of SO(n) and Sp(k) gauge groups. In particular, we get the mirror theory for SO(n) gauge group which is a completely new result. Furthermore, our proposal for the construction of the mirror theory predicts a nontrivial strong coupling limit of field theories with eight supercharges.

The contents of the paper are as follows. In section 2, we discuss some basic facts on Op-planes which will set the stage for calculating the mirrors. These include the four kinds of O3-planes and the s-configuration involving 1/2NS-brane and 1/2D5-brane. In section 3 we discuss the splitting of physical D5-branes on O3-planes. It is a crucial ingredient in our construction of mirror theory. By S-duality, we get the rules for how a physical NS-brane can split into two 1/2NS-branes or conversely how two 1/2NS-branes can combine into a physical NS-brane. The latter predicts a nontrivial transition of strongly coupled field theories. After these preparations, we give the mirror theory of a single gauge group with some flavors: Sp(k) in section 4, Sp$'$(k) in section[3] 5, SO($2k$) in section 6 and SO($2k+1$) in section 7. In sections eight and nine we generalize the mirror construction to products of two gauge groups: Sp(k) $\times$ SO($2m$) in section 8 and Sp$'$(k) $\times$ SO($2m+1$) in section 9. Finally, we give conclusions in section 10.

2. Some facts concerning O3-planes

In this section, we summarize some facts about the O3-plane which will be useful for the mirror construction later.

2.1 The four kinds of O3-planes

There are four kinds of O3-planes which we will meet in this paper (for a more detailed discussion, see [17]): $O3^+, O3^-, \widetilde{O3^+}, \widetilde{O3^-}$. However, before entering the specific discussion of $O3$-planes let us start from general Op-planes. When $p \leq 5$, there exist four kinds of orientifolds $Op^+, Op^-, \widetilde{Op^+}, \widetilde{Op^-}$. Among these four we are very familiar with $Op^+, Op^-, \widetilde{Op^-}$. They can be described perturbatively as the

[3] There are two ways to get Sp(k) gauge group: by $O3^+$-plane or $\widetilde{O3^+}$-plane. We denote the theory given by $O3^+$-plane as Sp(k) and the theory given by $\widetilde{O3^+}$-plane as Sp$'$(k).

JHEP11(2000)033

(b,c)	notation	charge	Gauge group	(b,c) after S-duality($p=3$ only)
$(0,0)$	O^-	-2^{p-5}	SO$(2n)$	(0,0) O^-
$(0,1)$	$\widetilde{O^-}$	$\frac{1}{2}-2^{p-5}$	SO$(2n+1)$	(1,0) O^+
$(1,0)$	O^+	2^{p-5}	Sp(n)	(0,1) $\widetilde{O^-}$
$(1,1)$	$\widetilde{O^+}$	2^{p-5}	Sp$'(n)$	(1,1) $\widetilde{O^+}$

Table 1: The summary of the properties of the four Op-planes. The charge is in units of physical Dp-brane.

fixed planes of the orientifold projection Ω which acts on the world sheet as well as the Chan-Paton factors. By different choices of the action Ω on the Chan-Paton factors we get two kinds of projections which we denote as $\pm$ projection. In the $+$ case, we can put only an even number of 1/2Dp-branes and the corresponding plane is the Op^+ plane. In the $-$ case, we can put an even or odd number of 1/2Dp-branes and the corresponding plane is Op^- for even number of 1/2Dp-branes and $\widetilde{Op^-}$ for odd number of 1/2Dp-branes. For $\widetilde{Op^-}$, because there is an odd number of 1/2Dp-branes, one 1/2Dp-brane must be stuck on the orientifold plane so that sometimes we consider the $\widetilde{Op^-}$ as the bound state of the Op^- and the 1/2Dp-brane (for more detailed discussion, the reader is referred to [18]). The $\widetilde{Op^+}$ is more complicated and is discussed in detail by Witten in [17]. In that paper, Witten observes $O3$-planes from a more unified point of view, namely discrete torsion (he deals with $O3$-planes. However the discussion can be easily generalized to other Op-planes). We can distinguish Op-planes by two Z_2 charges (b,c) with the definition $b=\int_{RP^2} B_{\rm NS}$ and $c=\int_{RP^{5-p}} C^{5-p}$ (the (b,c) is defined under modular two and the discussion presented here comes from lecture [19] already given by one of the authors at ITP, Santa Barbara; see also [20]). The second charge c exists only for $p\leq 5$. For $p>5$, it can not be defined and we are left only with two types of Op-planes (it is a little mysterious that $\widetilde{Op^-}$ does not exist for $p>5$, some arguments can be found in [19, 20]). We summarize the properties of these four Op-planes according the discrete torsions (b,c) in table 1 (where S-duality is applied only to $p=3$).

These four kinds of O-planes are not unrelated to each other and in fact change to each other when they pass through the 1/2NS-brane or 1/2D-brane [21, 10, 20]. The change is shown in figure 1: when $Op^-(\widetilde{Op^-})$ passes through the 1/2NS-brane, it changes to $Op^+(\widetilde{Op^+})$ and vice versa; when $Op^-(Op^+)$ passes through the 1/2D(p+2) -brane, it changes to $\widetilde{Op^-}(\widetilde{Op^+})$ and vice versa.

After the discussion of general Op-planes, we focus on $O3$-planes which will be used throughout this paper. For $O3$-planes, the charge of $O3^-$ is $-1/4$ while the charges of $O3^+,\widetilde{O3^-},\widetilde{O3^+}$ are 1/4. The fact that the charges for the latter three $O3$-planes are identical is not a coincidence and they are related to each other by the SL$(2,Z)$ duality symmetry in type IIB. In particular, under S-duality $O3^+$ and $\widetilde{O3^-}$ transform to each other while $\widetilde{O3^+}$ transforms to itself. $O3^-$ transforms to itself also under S-duality because it is the only O3-plane with $-1/4$ charge. One immediate

JHEP11(2000)033

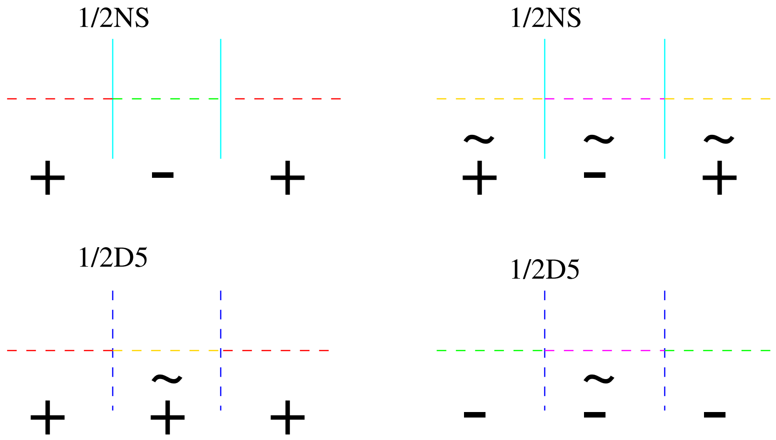


Figure 1: The change of four kinds of O3-planes as they cross 1/2NS-branes and 1/2D5-branes. In our brane setup, D3-brane and $O3$-plane will extend along X^{0126}, D5-brane, X^{012789} and NS-brane, X^{012345}. Henceforth, we use cyan (if the reader uses colored postscript rendering) lines to denote the 1/2NS-brane, blue lines to denote the 1/2D5-brane, dotted horizontal (red) lines to denote the $O3^+$-plane, dotted horizontal (green) lines to denote the $O3^-$-plane, dotted horizontal (yellow) lines to denote the $\widetilde{O3}^+$ and finally dotted horizontal (pink) lines to denote the $\widetilde{O3}^-$. Furthermore, for simplicity, we use $-, +, \tilde{-}, \tilde{+}$ to denote $O3^-, O3^+, \widetilde{O3}^-, \widetilde{O3}^+$ respectively.

application of the above S-duality property is that the change of O3-planes crossing the 1/2NS-brane is exactly S-dual to the change of O3-planes crossing the 1/2D5-brane. So our rule is consistent. The above discussions will be useful later in the study of mirror symmetry.

2.2 The supersymmetric configuration

In the procedures involved in mirror transformations, we need to break the D3-branes between the NS-brane and D5-brane to avoid the so called s-rule [7]. Furthermore, to read out the mirror theory from the brane setup it is convenient to move a 1/2NS-brane along the X^6 direction (our notations and conventions for the brane setups for all kinds of branes is given in the caption of figure 1.) to pass through the 1/2D5-brane such that the D3-branes ending on the 1/2NS-branes are annihilated in order to keep the linking number between 1/2NS-brane and 1/2D5-brane invariant. All these actions require the understanding of supersymmetric configurations in the presence of O3-planes. We summarize these results in this subsection. The tool in our discussion of s-configuration is still the conservation of linking number between 1/2NS-brane and 1/2D5-brane. The formula of linking number for 1/2NS-brane and 1/2D5-brane [7] is

$$
\begin{aligned}
L_{\rm NS} &= \frac{1}{2}(R_{\rm D5} - L_{\rm D5}) + (L_{\rm D3} - R_{\rm D3}) \\
L_{\rm D5} &= \frac{1}{2}(R_{\rm NS} - L_{\rm NS}) + (L_{\rm D3} - R_{\rm D3}) \, ,
\end{aligned}
\tag{2.1}
$$

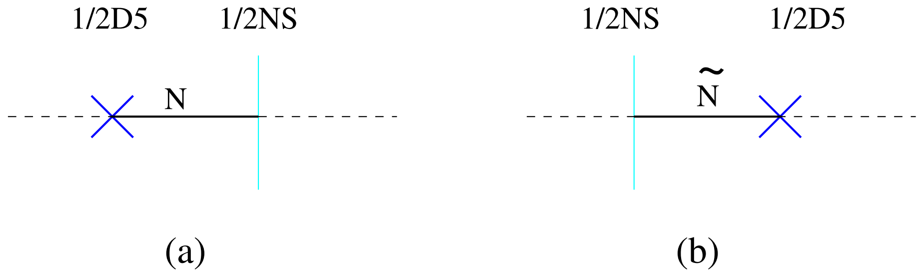


Figure 2: Starting from any one (left or right figure) , we move the 1/2NS-brane along X^6 direction to pass 1/2D5-brane and get the other (right or left). To allow such a process, we must conserve the linking number with the condition that $N, \tilde{N} \geq 0$. In this figure and henceforth, when NS-brane and D5-brane show at the same time in the figure with proper two-dimensional coordinates (for example, here $X = X^6, Y = X^5$), for clarification we use a line to denote an extended brane in these coordinates and use a cross to denote a point-like brane.

where $R_{\rm D5}$ ($L_{\rm D5}$) is the D5-charge to the right (left) of NS-brane (1/2D5-brane has 1/2 charge) and similar definition to others. Because we have four kinds of O3-planes we will have four kinds of supersymmetric configurations including one 1/2-NS brane and one 1/2-D5 brane. These four different cases are:

$$\begin{array}{lllll} (1) & O3^- \ (1/2\mathrm{D5} - 1/2\mathrm{NS}) & \mathrm{or} & (1/2\mathrm{NS} - 1/2\mathrm{D5}) \ \widetilde{O3^+} \,, \\ (2) & O3^+ \ (1/2\mathrm{D5} - 1/2\mathrm{NS}) & \mathrm{or} & (1/2\mathrm{NS} - 1/2\mathrm{D5}) \ \widetilde{O3^-} \,, \\ (3) & \widetilde{O3^-} \ (1/2\mathrm{D5} - 1/2\mathrm{NS}) & \mathrm{or} & (1/2\mathrm{NS} - 1/2\mathrm{D5}) \ O3^+ \,, \\ (4) & \widetilde{O3^+} \ (1/2\mathrm{D5} - 1/2\mathrm{NS}) & \mathrm{or} & (1/2\mathrm{NS} - 1/2\mathrm{D5}) \ O3^- \,, \end{array} \tag{2.2}$$

where the configuration $O3^-$ (1/2D5 – 1/2NS) or (1/2NS – 1/2D5) $\widetilde{O3^+}$ means that the $O3^-$ plane is at the left, $\widetilde{O3^+}$ at the right. In the middle we put 1/2NS-brane and 1/2D5-brane according to the order 1/2D5-1/2NS from left to right (see part (a) of figure 2) or 1/2NS-1/2D5 (see part (b) of figure 2).

The general pattern for the above four supersymmetric configurations is shown in figure 2, where we assume the number of connected D3-branes (in physical units) from the left 1/2D5-brane to the right 1/2NS-brane is N and from the left 1/2NS-brane to the right 1/2 D5-brane is $\tilde{N}$. So a configuration to be supersymmetric is equivalent to the solution of $N, \tilde{N} \geq 0$ such that they conserve the linking number after crossing.

2.2.1 The first case: $O3^- - - - \widetilde{O3^+}$

In this case we start from the brane setup (a) of figure 2 with $O3^-$ plane at the left, $\widetilde{O3^-}$ plane in the middle and $\widetilde{O3^+}$ plane at the right. The linking numbers are $L_{1/2\mathrm{D5}} = \frac{1}{2}(1/2 - 0) + [(-1/4) - (1/4 + N)] = -N - 1/4$ and $L_{1/2\mathrm{NS}} = 1/2(0 - 1/2) + [(1/4 + N) - (1/4)] = N - 1/4$. Now we move the 1/2D5 along X^6 direction to pass through 1/2NS and get the (b) of figure 2 with $O3^-$ plane at the left, $O3^+$

plane in the middle and $\widetilde{O3^+}$ plane at the right. For the latter we have linking numbers as $L_{1/2\mathrm{D5}} = 1/2(0-1/2) + [(\tilde{N}+1/4) - (1/4)] = \tilde{N} - 1/4$ and $L_{1/2\mathrm{NS}} = \frac{1}{2}(1/2-0) + [(-1/4) - (\tilde{N}+1/4)] = -\tilde{N} - 1/4$. Comparing these two linking numbers we get

$$N = -\tilde{N}\,. \tag{2.3}$$

It is a highly constraining equation. For the supersymmetric configuration, the only solution is $N = \tilde{N} = 0$. This means that when we break the D3-brane to go to the Higgs branch, we *can not* put D3-brane between 1/2NS-brane and 1/2D5-brane in this orientifold configuration.

2.2.2 The second case: $O3^+ - - - \widetilde{O3^-}$

Starting from brane setup (a) of figure 2 with $O3^+$ at the left, $\widetilde{O3^+}$ in the middle and $\widetilde{O3^-}$ at the right, we find the linking numbers as $L_{1/2\mathrm{D5}} = \frac{1}{2}(1/2-0) + [(1/4) - (1/4+N)] = -N + 1/4$ and $L_{1/2\mathrm{NS}} = 1/2(0-1/2) + [(1/4+N) - (1/4)] = N - 1/4$. Again by moving the 1/2D5-brane to pass through 1/2NS-brane we get the brane setup as (b) with the middle O3-plane changed from $\widetilde{O3^+}$ in (a) to $O3^-$ in (b) (the left and right O3-plane are invariant under the motion). The linking numbers for the latter are $L_{1/2\mathrm{D5}} = \frac{1}{2}(0-1/2) + [(\tilde{N}-1/4) - (1/4)] = \tilde{N} - 3/4$ and $L_{1/2\mathrm{NS}} = \frac{1}{2}(1/2-0) + [(1/4) - (\tilde{N}-1/4)] = -\tilde{N} + 3/4$. From these relations we find the equation

$$-N + 1 = \tilde{N}\,. \tag{2.4}$$

So for a consistent supersymmetric configuration there are three solutions: $(N, \tilde{N}) = (0,1)$, $(1/2, 1/2)$, $(1,0)$.

2.2.3 The third case: $\widetilde{O3^-} - - - O3^+$

For the third case, we start from the brane setup (a) with $\widetilde{O3^-}$ at the left, $O3^-$ in the middle and $O3^+$ at the right. The linking numbers are $L_{1/2\mathrm{D5}} = \frac{1}{2}(1/2-0) + [(1/4) - (-1/4+N)] = -N + 3/4$ and $L_{1/2\mathrm{NS}} = \frac{1}{2}(0-1/2) + [(-1/4+N) - (1/4)] = N - 3/4$. Now we move the 1/2D5-brane to pass through 1/2NS-brane and get the brane setup (b) with $\widetilde{O3^+}$ in the middle. The linking numbers become $L_{1/2\mathrm{D5}} = \frac{1}{2}(0-1/2) + [(\tilde{N}+1/4) - (1/4)] = \tilde{N} - 1/4$ and $L_{1/2\mathrm{NS}} = \frac{1}{2}(1/2-0) + [(1/4) - (\tilde{N}+1/4)] = -\tilde{N} + 1/4$. By comparing these relations we have

$$-N + 1 = \tilde{N}\,. \tag{2.5}$$

So again there are three solutions: $(N, \tilde{N}) = (0,1)$, $(1/2, 1/2)$, $(1,0)$.

2.2.4 The fourth case: $\widetilde{O3^+} - - - O3^-$

For the last case we start from the brane setup (a) with $\widetilde{O3^+}$ at the left, $O3^+$ in the middle and $O3^-$ at the right. The linking numbers are $L_{1/2\mathrm{D5}} = \frac{1}{2}(1/2-0) + [(1/4) -$

$(1/4+N)] = -N+1/4$ and $L_{1/2\mathrm{NS}} = \frac{1}{2}(0-1/2)+[(1/4+N)-(-1/4)] = N+1/4$. Now we move the 1/2D5-brane to pass through 1/2NS-brane and get the brane setup (*b*) with $\widetilde{O3}^-$ in the middle. The linking numbers change to $L_{1/2\mathrm{D5}} = \frac{1}{2}(0-1/2)+[(\tilde{N}+1/4)-(-1/4)] = \tilde{N}+1/4$ and $L_{1/2\mathrm{NS}} = \frac{1}{2}(1/2-0)+[(1/4)-(\tilde{N}+1/4)] = -\tilde{N}+1/4$. From these relations we have

$$-N = \tilde{N}\,. \tag{2.6}$$

The only solution is $(N,\tilde{N}) = (0,0)$ as in the first case.

Let us summarize the results in the last four subsections. When the charge of $O3$-planes at the two sides are the same (case two and case three), the condition is $N+\tilde{N} = 1$, so there is annihilation or creation of D3-branes in crossing. When the charge of $O3$-planes at the two sides are different (case one and case four), the condition is $N = \tilde{N} = 0$, so there can not be any D3-branes between the 1/2NS-brane and 1/2D5-brane.

3. The splitting of the physical brane

To construct the mirror theory by brane setups, we can follow the procedure given in the introduction [7]. However, in the presence of the O3-plane, we need one new input: how to *split* the physical D5-brane into two 1/2D5-branes on the O3-plane. Initially, the physical D5-brane can be placed off the O3-plane in pairs of 1/2D5-branes (see figure 3). We can move the pair of 1/2D5-branes to touch the O3-plane. After touching the O3-plane, in principle every 1/2D5-brane can move freely on the O3-plane. We call such an independent motion of the 1/2D5-brane as "*splitting*" of the physical D5-brane. We want to emphasize that the splitting of a physical D5-brane into two 1/2D5-branes is a nontrivial dynamical process in string theory and can be applied to many situations. Here we need the splitting because in the mirror theory, the gauge theory is given by D3-branes ending on 1/2NS-branes which are the S-dual of 1/2D5-branes in the original theory. In this paper, we give only a preliminary discussion. We found some novel results: sometimes there is a creation of one physical D3-brane between these two 1/2D5-branes; sometimes there

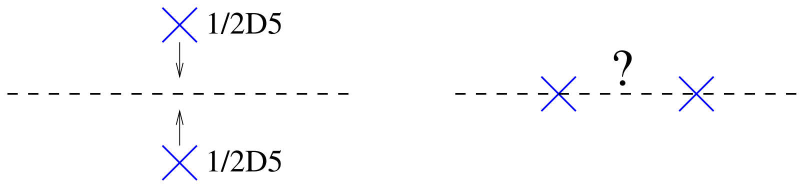


Figure 3: Splitting of a D5-brane on the O3-planes. The left figure shows that a pair of 1/2D5-branes moving to touch the O3-plane. The right figure shows that when they touch the O3-plane they can split. The ? in the middle of these two 1/2D5-branes means there is nontrivial dynamics dependent on different situations.

is an annihilation and sometimes, no creation and no annihilation. We found these results by matching the Higgs branch moduli of the Sp or SO theory with the correct dimension of moduli space.

3.1 The splitting of D5-branes without ending D3-branes

Before going to the general situation let us discuss the splitting of D5-branes which do not have any D3-branes ending on them. First we discuss the case where there is only one physical D5-brane and O3-plane (see figure 3). Before splitting, every 1/2D5-brane has linking number *zero*. After splitting, there can be N physical D3-branes between these two 1/2D5-branes (to keep supersymmetry, there can not be anti-D3-branes between them; furthermore, because here we do not have any D3-branes initially, there can not be annihilation either). Let us calculate the linking number after splitting:

O3 before splitting	ΔL_{left}	ΔL_{right}
$O3^+$	$-N$	N
$\widetilde{O3}^+$	$-N$	N
$O3^-$	$-1/2-N$	$1/2+N$
$\widetilde{O3}^-$	$1/2-N$	$-1/2+N$

(3.1)

In BPS states, we have the tension of D5-branes proportional to their charge (linking number). To have the minimum tension configuration, it is natural to have $N=0$ for the first three cases. However, for the last case, $N=0$ and $N=1$ are equally favorable just from the point view of tension. We will fix the ambiguity in the next paragraph. However, before we end this paragraph, we want to emphasize that no matter what case it is, the total change in linking number is always

$$\Delta L = 0 \qquad \text{or} \qquad \Delta L = \pm\frac{1}{2}\,.$$

We can fix the ambiguity for the last case by considering higgsing. Starting from the SO(3) gauge theory with one flavor, we can Higgs it to SO(2) with one singlet (there are $3-1=2$ gauge fields which get mass, so we leave only $3\times 1-2=1$ singlet). In part (a) of figure 4 we assume $N=0$ in the splitting process and go to the Higgs branch. By moving 1/2D5-branes outside we find the final theory is SO(2) without singlets in part (b). This means that our assumption is wrong. Choosing the other assumption $N=1$ in part (c), by moving 1/2D5-branes outside we get the final theory is SO(2) with a singlet in part (d) which is exactly what we expect from the field theory. This shows that, for matching the correct moduli dimensions of Higgs branch, in last case of (3.1) there should be a D3-brane created in the splitting.

The discussion of the splitting of physical D5-branes becomes more complex if there are more than one D5-brane to be split. The complexity manifests in the

JHEP11(2000)033

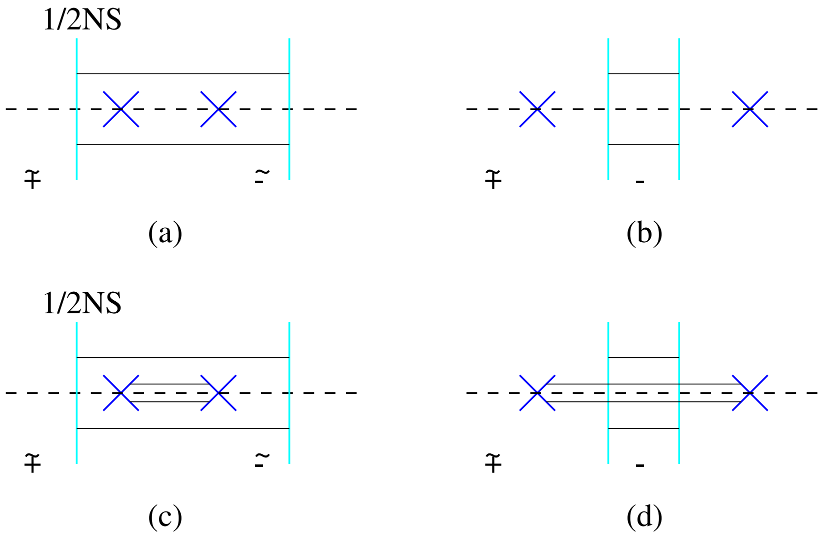


Figure 4: The Higgs branch of SO(3) with one flavor. (*a*) We assume when the splitting, there is no D3-brane generated. (*b*) By moving 1/2D5-branes outside, we get SO(2) without singlet. (*c*) We assume when the splitting, there is a D3-brane generated. (*d*) By moving 1/2D5-branes outside, we get SO(2) with one singlet given by D3-brane ending on 1/2D5-brane.

last two cases in (3.1) because in these cases there is a change of linking number ($\Delta L = \pm 1/2$) for every 1/2D5-brane. Before splitting, we have, for example, $2n$ 1/2D5-branes with linking number *zero*. After splitting, we have n 1/2D5-branes with linking number 1/2 and n, with linking number $-1/2$. The different order of linking number gives different physical content, i.e., the order determines when there should be D3-branes created and when there are no D3-branes created.

To illustrate our idea, let us see figure 5. After splitting one D5-brane according to the analysis in the last paragraph, we continue to split the second D5-brane. However, in this case, we have two choices. In the first choice, the second D5-brane is far away from the first D5-brane in X^6 direction like part (*a*). So locally the splitting should be the same as the first D5-brane and we get part (*b*). Notice that the order of linking number of 1/2D5-branes is $-1/2$, $+1/2$, $-1/2$, $+1/2$. In the second choice, the second D5-brane is in the middle of the pair of first 1/2D5-branes as part (*c*). Naively, the second D5-brane will see the $O3^-$-plane (in fact, D5-brane will see more) so the splitting looks like to go as part (*d*) with the order of linking number $-1/2$, $-1/2$, $+1/2$, $+1/2$. However, part (*d*) is not consistent with the Higgs branch of SO(3) with two flavors. Furthermore, because there are eight supercharges, the different positions of D5-branes should not effect the physics. So we argue that from part (*c*) we should get part (*b*) too. In part (*c*), the second D5-brane sees not only the $O3^-$-plane, but also the one created D3-branes, 1/2D5-branes at left with $\Delta = -1/2$ and 1/2D5-branes at right with $\Delta = +1/2$. This more complete information determines that the second D5-brane will split to part (*b*).

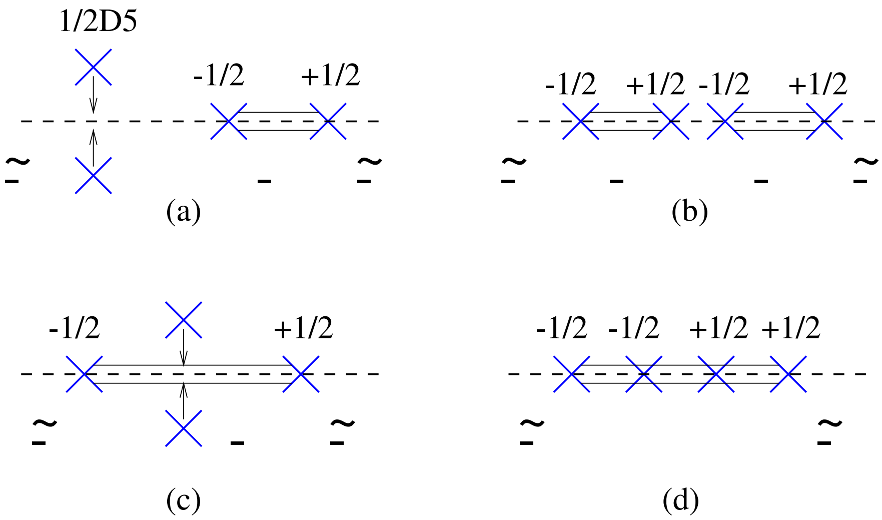


Figure 5: The splitting of the second D5-brane. (a) second D5-brane is far away from the first D5-brane. (b) the splitting of second D5-brane from configuration in part (a). (c). second D5-brane is in the middle of first D5-brane. (d) the naive splitting of second D5-brane which turns out to be wrong.

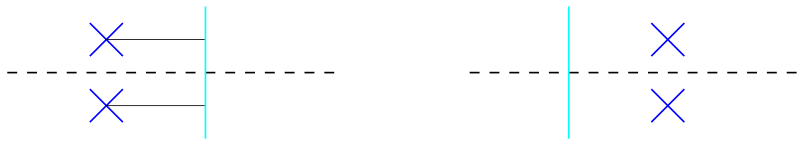
From the above observation, we propose that the correct order of linking number should be $-1/2$, $+1/2$, $-1/2$, $+1/2$, $\ldots$, $-1/2$, $+1/2$ (notice the alternating fashion of $-1/2$ and $+1/2$). We make such a suggestion because it is the only correct order which can produce the consistent Higgs pattern for SO(K) gauge group with N flavors. It will be very interesting if we can derive such a rule from string theory. Furthermore, this proposal will give very interesting predictions which we will discuss later.

Let us pause a moment to summarize the results we have obtained above. Without the D3-brane ending on D5-branes, (1) the change of linking number of 1/2D5-branes is $\Delta L = 0$ for $O3^+, \widetilde{O3^+}$ and $\Delta L = \pm 1/2$ for $O3^-, \widetilde{O3^-}$; (2) for the splitting of a bunch of D5-branes, the order of linking number is $-1/2$, $+1/2$, $-1/2$, $+1/2$, $\ldots$, $-1/2$, $+1/2$.

3.2 The splitting of D5-branes with ending D3-branes

After the discussion of the splitting of D5-branes without D3-branes ending on them, we consider the case that there are N D3-branes ending on them. The results for this latter case can be derived from the results in the last subsection. For example, let us discuss the case of one D5-brane with one ending D3-brane in figure 6. We can add one 1/2NS-brane such that the D3-brane ending on it as part (a). Then we can move D5-brane to the right of 1/2NS-brane and annihilate the D3-brane as part (b). Now the part (b) is the case we discussed in the last subsection. We can split the physical D5-brane and move two 1/2D5-branes to left of 1/2NS-brane by using the result in section 2. By this loop, we finally get the splitting of D5-brane with

JHEP11(2000)033



(a) (b)

Figure 6: (a) One D3-brane ends on a physical D5-brane. We can add a 1/2NS-brane at the right. It should not affect the discussion. (b) By moving D5-brane to right of 1/2NS-brane we annihilate the ending D3-brane.

O3-plane	N = even	N = odd
$O3^+$	$\Delta L = 0$	$\Delta L = \pm 1/2$ in order of $-1/2, +1/2, -1/2, +1/2, \ldots$
$\widetilde{O3}^+$	$\Delta L = 0$	$\Delta L = \pm 1/2$ in order of $-1/2, +1/2, -1/2, +1/2, \ldots$
$O3^-$	$\Delta L = \pm 1/2$ in order of $-1/2, +1/2, -1/2, +1/2, \ldots$	$\Delta L = 0$
$\widetilde{O3}^-$	$\Delta L = \pm 1/2$ in order of $-1/2, +1/2, -1/2, +1/2, \ldots$	$\Delta L = 0$

Table 2: The rules of splitting of D5-brane, where $N = |N_L - N_R|$ is the difference of D3-branes ending on D5-brane from the left and the right.

one ending D3-brane. For more D3-branes ending on D5-branes we can add more 1/2NS-brane and repeat the above procedure.

Although the above trick solves our problem completely, it is too tedious and we need a more direct way to see it. Notice that the change of the linking number of 1/2D5-branes happens only at splitting. So we can use the changing of linking number as the rule to determine the splitting of D5-brane. In general there will be N_L D3-branes ending on D5-brane from the left and N_R D3-branes, from right. The rule depends only on the absolute difference between N_L, N_R, i.e., $N = |N_L - N_R|$. We summarize the rule in table 2.

3.3 The splitting of NS-branes and novel predictions of field theory in the strong coupling limit

Making S-duality, we can get the rules of splitting physical NS-branes into 1/2NS-branes on O3-plane as table 3.

From table 3, we get two predictions of $N = 4$ three dimensional field theory in the strong coupling limit (see figure 7). In the first case (part (a) of figure 7), the field theory is $\mathrm{SO}(2k) \times \mathrm{Sp}(k) \times \mathrm{SO}(2k)$ with two bifundamentals. From the brane setup in part (a), we see that, by reversing the process of the splitting of the NS-brane, we can move two middle 1/2NS-branes to meet together and leave

O3-plane	N = even	N = odd
$\widetilde{O3}^{-}$	$\Delta L = 0$	$\Delta L = \pm 1/2$ in order of $-1/2, +12, -1/2, +1/2, \ldots$
$\widetilde{O3}^{+}$	$\Delta L = 0$	$\Delta L = \pm 1/2$ in order of $-1/2, +1/2, -1/2, +1/2, \ldots$
$O3^{-}$	$\Delta L = \pm 1/2$ in order of $-1/2, +1/2, -1/2, +1/2, \ldots$	$\Delta L = 0$
$O3^{+}$	$\Delta L = \pm 1/2$ in order of $-1/2, +1/2, -1/2, +1/2, \ldots$	$\Delta L = 0$

Table 3: The rules of splitting of NS-brane, where $N = |N_L - N_R|$ is the difference of D3-branes ending on NS-brane from the left and the right.

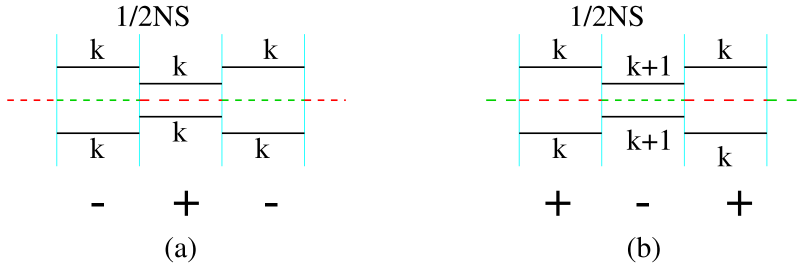


Figure 7: (a) SO($2k$) $\times$ Sp(k) $\times$ SO($2k$) gauge theory with two bifundamentals. (b) Sp(k) $\times$ SO($2k+2$) $\times$ Sp(k) gauge theory with two bifundamentals.

$O3^{-}$-plane. In field theory, moving two middle 1/2NS-branes together corresponds to the strong coupling limit of Sp($2k$) gauge theory, and moving NS-brane off the $O3^{-}$-plane corresponds to turning on *"FI-parameters"*.[4] So our brane configuration predicts that, at the strong coupling limit of Sp($2k$) and the turning of FI-parameter, the original theory SO($2k$) $\times$ Sp(k) $\times$ SO($2k$) with two bifundamentals will flow to SO($2k$) without any flavor. The second case is given in part (b) of figure 7. By the similar arguments, we predict that at the strong coupling limit of SO($2k+2$) and the turning of FI-parameter, the field theory Sp(k) $\times$ SO($2k+2$) $\times$ Sp(k) with two bifundamentals will flow to Sp(k) without any flavor. It will be interesting to check these two predictions from the field theory point of view.

4. The mirror of Sp(k) gauge theory

Now we start to construct mirror pairs using the above knowledge. First let us discuss Sp(k) gauge theory with N fundamental flavors. In this case, the brane setup is as follows: we put $2k$ 1/2D3-branes, i.e., branes and their images under the O-plane (extended in X^{0126}) ending on two 1/2NS branes (extended in X^{012345})

[4]In fact, it is a hidden *"FI-parameters"*. We will discuss it more in section 4.3.

along X^6 while $2N$ 1/2D5-branes (extended in X^{012789}) are put in the middle (see (a) of figure 8). Then $O3$-planes from the left to right read as $O3^-, O3^+, O3^-$. This $O3$-plane configuration reminds us of the special s-configuration discussed in the last section. Because in the presence of $O3$-planes the s-configuration is a little different from the known s-rule in [7], we will demonstrate the detailed steps for the mirror construction for Sp(1) with three flavors. Thereafter we quickly go to the general Sp(k) case.

4.1 Sp(1) with the 3 flavors

For the Sp(1) gauge theory with 3 flavors we have the following information about the moduli space of the Higgs branch and the Coulomb branch as well as the FI-parameters and mass parameters:

$$\begin{aligned} d_v &= 1 \\ d_H &= 3 \times 2 - 3 = 3 \\ \#m &= 3 \\ \#\zeta &= 0\,. \end{aligned} \tag{4.1}$$

After the mirror map, we should have a mirror theory which has $d_v = 3$, $d_H = 1$, $\#\zeta = 3$, $\#m = 0$, i.e. the Coulomb branch and the Higgs branch are interchanged while the mass parameters and FI-parameters are exchanged [1]. However, when we count these parameters, sometimes we meet nontrivial situations, such as the "hidden FI-term" explained in [9]. We will see later that these "hidden parameters" arise in our construction and will discuss them in more detail later.

The details of the mirror construction are given in figure 8. Let us go step by step. Part (a) is just the brane setup for Sp(1) with three fundamental flavors. By moving the physical D5-brane to touch the orientifold $O3^+$ plane, i.e., setting the masses to zero, we can split them into 1/2D5-branes as in part (b). Now we go to the Higgs branch by splitting the D3-branes between those 1/2NS-branes and 1/2D5-branes. However, from (2.3) and (2.6), we must split these D3-branes as given by part (c). The crucial point is that there is no D3-brane connected between the 1/2NS-brane and its nearest 1/2D5-brane because it is prohibited by the supersymmetric configuration discussed in section 2. Now we can use the rules (2.3) and (2.6) to move the left 1/2NS-brane crossing the neighboring right 1/2D5-brane and the right 1/2NS-brane crossing the neighboring left 1/2D5-brane. The result is given by part (d). Notice that in such a process, no D3-brane is created or annihilated. Applying (2.4) and (2.5) to move the 1/2NS-brane across 1/2D5-brane, we reach part (e). In this process, the physical D3-brane which connects the 1/2NS-brane and 1/2D5-brane is annihilated. Now we can apply the mirror transformation to give the result shown in part (f). However, it is a little hard to read out the final gauge theory because of

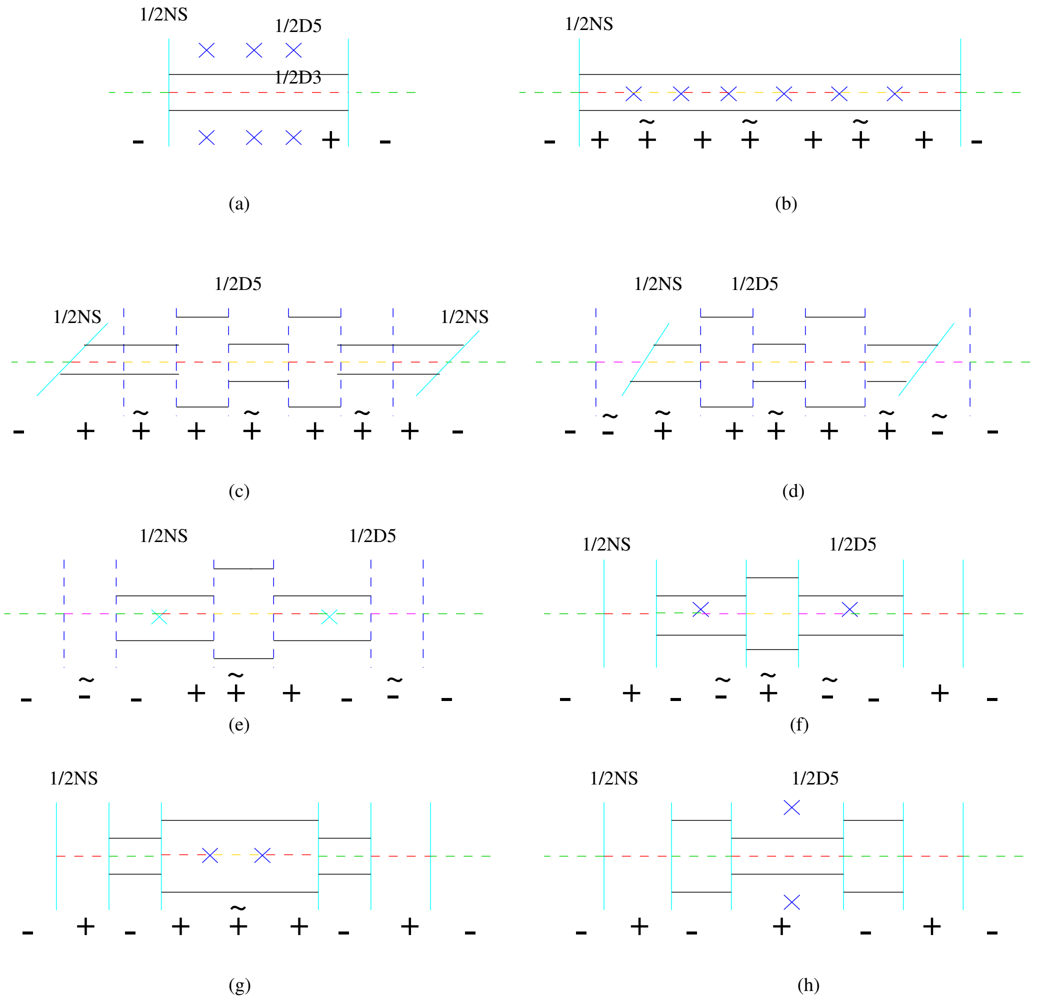


Figure 8: The detailed steps for getting the mirror of Sp(1) with three fundamental flavors. (a) The brane setup. (b) Splitting of the physical D5-branes. (c) The Higgs branch obtained by splitting the D3-brane. Notice the special splitting of these D3-branes. (d) Using the result of supersymmetric configuration we can move 1/2NS-brane one step inside. (e) Using again the rule of supersymmetric configuration we move the 1/2NS-brane one further step inside. In this step, the D3-brane ending on the 1/2NS-brane is annihilated. (f) S-dual of part (e). (g) However, we can not read out the final gauge theory from the brane setup in (f). For avoiding the ambiguity, we can move 1/2NS-brane one further step inside. (h) A special property of our example is that we can combine two 1/2D5-branes in part (g) together and leave the $O3^+$ plane.

the $O3^-$ and $\widetilde{O3}^-$ projections in the same interval. We can get rid of this ambiguity by applying (2.3) and (2.6) again to reach the result in part (g).

Now we have the brane setup for the mirror theory in part (g) of figure 8. We can read out the theory directly from the brane setup according the standard rule: For $2k$ 1/2D3-brane stretching between two 1/2NS-branes with $O3^-$, $\widetilde{O3}^-$, $O3^+$, $\widetilde{O3}^+$ planes we get SO$(2k)$, SO$(2k+1)$, Sp(k), Sp$'(k)$ gauge groups respectively. For one 1/2D5-

JHEP11(2000)033

brane between two 1/2NS-branes it contributes one fundamental half-hypermultiplet for that gauge group. For one physical D5-brane between two 1/2NS-branes it contributes one fundamental hypermultiplet for the gauge group. For two gauge groups which have a common 1/2NS-brane there is a bifundamental (in the presence of $O3$-plane, such bifundamental is, more exactly, half-hypermultiplet). Applying the above rules we immediately get the mirror theory as $\mathrm{SO}(2)\times\mathrm{Sp}(1)\times\mathrm{SO}(2)$ with two bi-fundamentals and one fundamental for Sp(1). Here we want to emphasize that in general we get only half fundamental hypermultiplets coming from the 1/2D5-brane. The unusual point for this explicit example is that the two 1/2D5-branes are in the same interval such that they can combine together and leave the orientifold (see section 3). Now let us calculate the moduli spaces and parameters to see if they are really mirror to each other. For the mirror theory in the part (h) of figure 8, it is easy to get the dimensions of moduli spaces as $d_v = 1+1+1 = 3$ and $d_H = (2\times 2+2\times 2)/2+1\times 2-(1+1+3) = 6-5 = 1$, so we see the results match when comparing to (4.1). However, when we turn to calculate the mass parameters and FI-parameters, a mismatch occurs. In the mirror theory, we have two bifundamentals and one fundamental. For the two bifundamentals we do not know how to turn the mass parameters so we get the $\#m = 1$. Because there are no U(1) factors in the mirror theory, it seems that we should get $\#\zeta = 0$. Now comparing with the original theory, we find a mismatch in the mass parameters and FI-parameters. The solution of the above mismatch is given by the concept of "hidden FI-term" which we will discuss later [9].

4.2 Another method to go to the Higgs branch

In the above procedure, we split D5-branes first, then went to the Higgs branch by splitting the D3-branes. However, we can go to the Higgs branch in another way by splitting the D3-brane first on the physical D5-brane and then splitting the D5-brane on the O3-plane. The procedure of this second method is drawn in figure 9. In part (a) , we keep the D5-branes off the O3-plane and split the D3-branes to go the Higgs branch (such splitting is very familiar to us already, see [7]). By moving the physical D5-brane to cross the 1/2NS-brane, we can get rid of the D3-brane ending one D5-brane and 1/2NS-brane. The result is shown in part (b). Now we move the D5-brane to the O3-plane and split them. For consistency with the first method in the last subsection we must require the splitting of D5-brane with one D3-brane ending on it as the rule given in section 3.2. In fact, as we discussed above, we find all rules in section 3.2 in this way. It is easy to check that in this example we should get the same result as part (e) of figure 8.

4.3 The "hidden FI-term"

We have met the mismatch of mass and FI parameters in the above mirror pair. It is time for us to talk more about it in this subsection. In fact, such a mismatch of mass

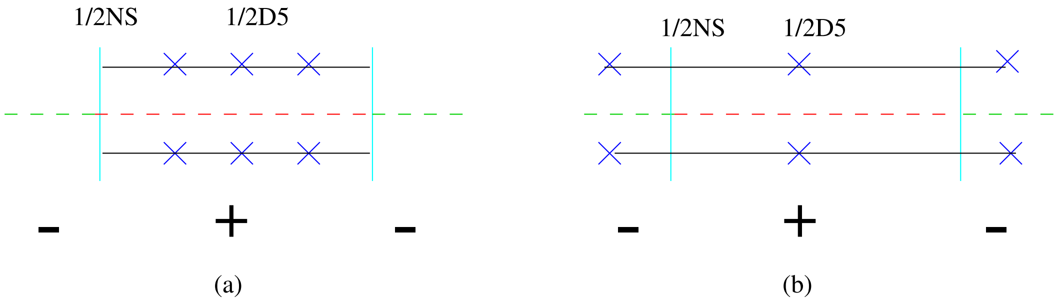


Figure 9: The other method to go to the Higgs branch: (a) The incomplete Higgs branch when D5-branes are off the O3-plane. (b) By moving 1/2NS-brane one step inside, we get rid of the D3-brane ending on 1/2NS-brane.

and FI parameters in mirror pair is not new to us. Kapustin found this problem in [9]. In that paper, he considers the mirror of Sp(k) with an antisymmetric tensor and n fundamental flavors. He found that when $n = 2, 3$ the quivers of the mirror theory are in fact affine A_1 for $n = 2$ and affine A_3 for $n = 3$. However, it is a well-known fact that a gauge theory given by an affine A_n quiver has one mass parameter. On the other hand, classically the original Sp(k) theory does not have any FI parameters. Kapustin suggests the concept of *"hidden FI term"* to resolve the conflict. Such a term arises as the deformation in the infrared limit and has the same quantum number as a FI-term. Because it is a quantum effect, these deformations need not have a lagrangian description in the ultraviolet. To count the number of hidden FI deformation we simply count the mass parameters in the mirror theory. Now applying Kapustin's explanation to our example, we find there is one "hidden FI-term" for the original theory and three "hidden FI-terms" for the mirror theory. This result is consistent with Kapustin's result. Notice that for $k = 1$ the antisymmetric tensor of Sp(k) does not exist, so his theory is in exact agreement with our original theory and we both find one "hidden FI-term". Hereafter we do not discuss the matching of the mass and FI parameters anymore, but we will mention the case when there exists a "hidden FI-term" for the original theory.

The appearance of the *"hidden FI term"* indicates another important aspect of the possible enhanced hidden global symmetry. In [1], the authors observed that the fixed point can have global symmetries which are manifest in one description but hidden in another (i.e., can be seen only quantum mechanically). For example, the U(1) with two flavors is a self-mirror theory. On a classical level we have SU(2) $\times$ U(1) global symmetry, where SU(2) is the flavor symmetry and U(1) is the global symmetry connecting one FI-parameter (FI-parameter can be considered as a component in the background vector supermultiplet of U(1)). However, at the fixed point, the U(1) global symmetry is enhanced to SU(2). This enhanced symmetry can be easily seen in the brane setup of the mirror theory because in this special case

(U(1) with two flavors), the two D5-branes (the S-dual of two NS-branes in original symmetry) meet in same interval. This is another advantage of brane setup because we can see a lot of nontrivial phenomena pictorially. In later sections, when we find the case where there is a *"hidden FI term"*, we will also discuss the enhanced global symmetry.

There is another interesting aspect which is worth mentioning. If our construction is right, it seems that we have two different theories which are mirror to the same one because in [9, 10] we can construct the mirror of Sp(k) gauge theory by using the $O5^-$ plane. This is also met by Kapustin in [9]. He noticed that two theories, (1) the Sp(k) gauge theory with an antisymmetric tensor plus two or three fundamental and (2) the U(k) gauge theory with an adjoint plus two or four fundamental flavors, are mirror to the same affine A_1 or A_3 quiver theory. Because mirror symmetry is a property in the infrared limit of gauge theory, such a non-uniqueness is allowed. Actually the brane picture provides a definition of the theory beyond the infrared limit and the non-uniqueness can be seen in nature by having two different brane representations of the same field theory.

4.4 Sp(2) with 6 flavors

With the experience of Sp(1) gauge theory, we can deal with the Sp(2) with 6 flavors very quickly. The moduli spaces for the original theory have $d_v = 2$ and $d_H = 6\times4-10 = 14$ (as mentioned above, in the following discussion we do not discuss the issue of mass parameters and FI-parameters). The steps for getting the mirror theory is in figure 10. From the brane setup (d) in figure 10 we read out that the mirror theory as $\mathrm{SO}(2)\times\mathrm{Sp}(1)\times\mathrm{SO}(4)\times\mathrm{Sp}(2)\times\mathrm{SO}(5)\times\mathrm{Sp}(2)\times\mathrm{SO}(4)\times\mathrm{Sp}(1)\times\mathrm{SO}(2)$ with 8 bifundamentals and two fundamental half-hypermultiplets one for each Sp(2) gauge theory. By an easy calculation, we can check the moduli spaces as: $d_v = 4\times1+5\times2 = 14$ and $d_H = (2\times4+2\times8+2\times16+2\times20+2\times4)/2-(2+2\times3+2\times6+3\times10) = 52-50 = 2$.

4.5 The general case

Now we discuss the general case, i.e., Sp(k) with N fundamental flavors (to get the complete higgsing, we have to assume that $N \geq 2k$). The moduli space has $d_v = k$ and $d_H = 2kN - k(2k+1)$. The steps for getting the mirror theory are shown in figure 11.

From it we can read out that the mirror theory are $\mathrm{SO}(2)\times\mathrm{Sp}(1)\times\mathrm{SO}(4)\times\mathrm{Sp}(2)\times\cdots\times\mathrm{Sp}(k-1)\times\mathrm{SO}(2k)\times(\mathrm{Sp}(k)\times\mathrm{SO}(2k+1))^{n-2k-1}\times\mathrm{Sp}(k)\times\mathrm{SO}(2k)\times\mathrm{Sp}(k-1)\times\cdots\times\mathrm{Sp}(1)\times\mathrm{SO}(2)$ with bifundamentals and one fundamental half-hypermultiplet for each the first and the last Sp(k) gauge groups. For clarity, the corresponding quiver diagram of the above mirror theory is also drawn in part (c) of

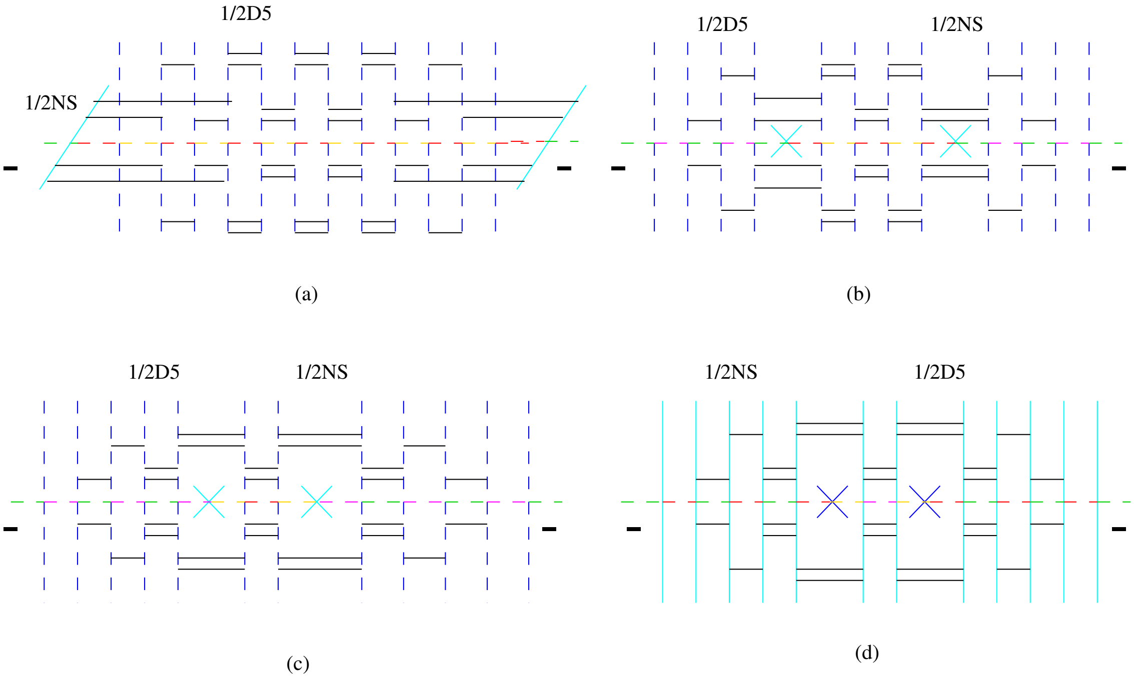


Figure 10: (a) The Higgs branch of Sp(2) with 6 flavors. Notice how we split the D3-branes according the supersymmetric configuration. (b) By moving 1/2NS-brane across the 1/2D5-brane, we get rid of the D3-brane ending on 1/2NS-brane. (c) However, to read out the correct mirror theory, we need to move the 1/2NS-brane one step further inside. (d) By S-duality of part (c) we get the brane setup of the mirror theory.

this figure. Now we can calculate the moduli spaces of the mirror as

$$
\begin{aligned}
d_v &= 4\sum_{n=1}^{k-1} n + (2N-4k+1)k = 2Nk - k(2k+1) \\
d_H &= \left[\frac{1}{2}\times 2\sum_{n=1}^{k-1}((2n)^2 + 2n(2n+2)) - \right. \\
&\quad \left. + \frac{1}{2}\left(2(2k)^2 + (2N-4k-2)2k(2k+1)\right) + 2k\right] - \\
&\quad - \left[2\sum_{n=1}^{k-1}(n(2n-1)+n(2n+1)) + (2N-4k-1)k(2k+1) + 2k(2k-1)\right] \\
&= k.
\end{aligned} \tag{4.2}
$$

As mentioned above, for general N, k we get only half-hypermultiplets coming from the 1/2D5-branes in the mirror theory. However, there are two degenerate cases where one fundamental hypermultiplet does exist instead of two half-hypermultiplets. The first case is when $N = 2k$. In this case, we do not need to move the 1/2NS brane further from part (a) to part (b) in figure 11. Instead, we can make the mirror transformation directly from part (a). In the mirror theory, we get only one SO$(2k)$ gauge

JHEP11(2000)033

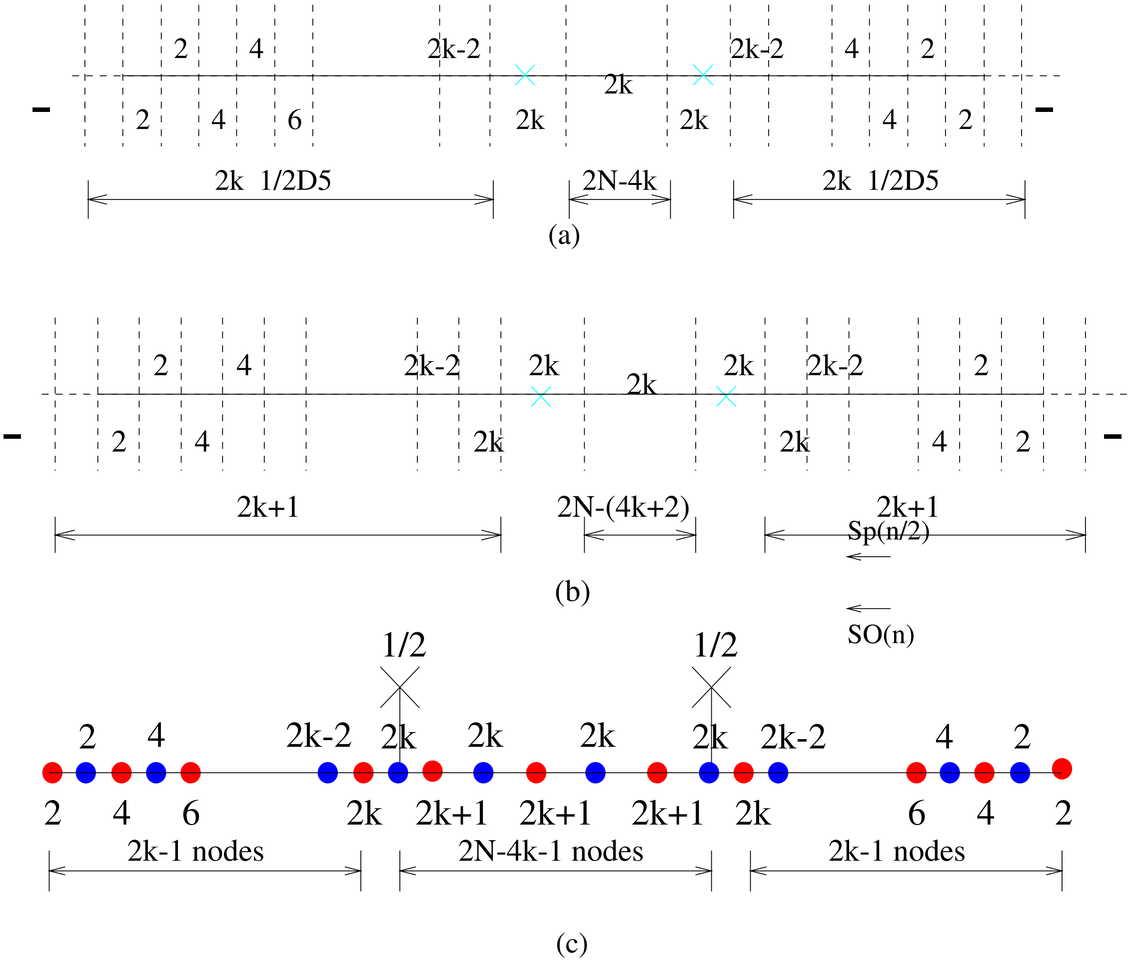


Figure 11: The mirror of $\mathrm{Sp}(k)$ gauge theory with N flavors. Because of the complexity, in this figure we do not keep track of the change of O3-plane anymore and use dotted horizontal black line to express all O3-planes. However, we do keep track of the intervals which give the Sp or SO group in the final mirror theory by using the number above the O3-plane to denote the Sp group and below to denote SO group. (a) The Higgs branch of $\mathrm{Sp}(k)$ with N flavors. Notice that the pattern of the number of 1/2-D3 branes between two nearby 1/2D5-branes is, from left to right, $0, 2, 2, 4, 4, 6, \ldots 2k-2, 2k^{2N-4k+1}, 2k-2, 2k-2, \ldots, 4, 4, 2, 2, 0$. (b) To read out the mirror theory in general we need to move the 1/2NS-brane one step further inside. However, we can consider (b) as the brane setup of the mirror theory too by just thinking of the dotted vertical line as 1/2NS brane and the cross as 1/2D5-brane. (c) For convenience, we draw the quiver diagram. We use red dots for SO groups and blue dots for Sp groups. We also write the number above for an Sp group and under for an SO group.

group but with one flavor for this $\mathrm{SO}(2k)$. As explained above, such a flavor hints a "hidden FI-term" in the original theory. The second case is when $N = 2k+1$, where we get only one $\mathrm{Sp}(k)$ gauge group in mirror theory, but also with one flavor of the $\mathrm{Sp}(k)$ which also suggests a "hidden FI-term" in the original theory. For $k = 1$, the two cases where a "hidden FI-term" shows is given in [9]. For $k \geq 2$ it is a new result.

As we mentioned in section 4.3, in the case where a "hidden FI-term" shows we should consider the possible enhancement of global symmetry. In general the theory has global $\mathrm{SO}(2N)$ flavor symmetry. When $N = 2k$, the global symmetry will be enhanced to $\mathrm{SO}(2N) \times \mathrm{Sp}(1)$. The factor $\mathrm{Sp}(1)$ can be seen from the mirror theory, where two 1/2D5-branes meet and give one flavor to the $\mathrm{SO}(2k)$ gauge group (notice the flavor symmetry for $\mathrm{Sp}(k)$ gauge groups is $\mathrm{SO}(2N)$, for $\mathrm{Sp}'(k)$ gauge groups, $\mathrm{SO}(2N+1)$, for $\mathrm{SO}(2k)$ gauge groups, $\mathrm{Sp}(N)$ and for $\mathrm{SO}(2k+1)$ gauge groups, $\mathrm{Sp}'(N)$). When $N = 2k+1$, the global symmetry will be enhanced to $\mathrm{SO}(2N) \times \mathrm{SO}(2)$ because in this case, the one extra flavor in mirror theory belongs to the $\mathrm{Sp}(k)$ gauge group.

5. The mirror of $\mathrm{Sp}'(k)$ gauge theory

We know that the $O3^+$ and $\widetilde{O3^+}$ projections both give $\mathrm{Sp}(k)$ gauge theory. To distinguish them, we denote the gauge theory given by $O3^+$ projection as $\mathrm{Sp}(k)$ and that by $\widetilde{O3^+}$ as $\mathrm{Sp}'(k)$. After the discussion of the mirror of $\mathrm{Sp}(k)$ gauge group in the last section, we now address the $\mathrm{Sp}'(k)$ case in this section. The brane setup of Sp' is just to replace the $O3^\pm$ in $\mathrm{Sp}(k)$ by $\widetilde{O3^\pm}$ (for example, see figure 12). However, by such a replacement, the theory becomes $\mathrm{Sp}(k)$ gauge theory with n flavors plus two half-hypermultiplets contributed from the $\widetilde{O3^-}$ at the two sides (notice that $\widetilde{O3^-}$ can be considered as $O3^-$ plus a 1/2D3-brane). We will start the discussion also from a simple example, then go to the general case. Furthermore, we will compare the mirror of $\mathrm{Sp}(k)$ and $\mathrm{Sp}'(k)$ and show that in fact they give the same mirror theory.

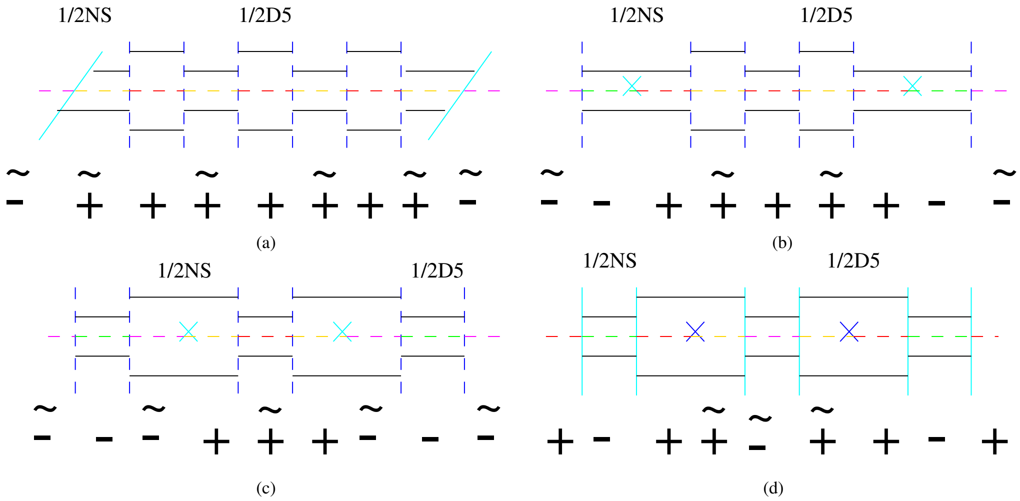


Figure 12: (a) The Higgs branch of $\mathrm{Sp}'(1)$ with three flavors. Notice how we split D3-branes to satisfy the supersymmetric configuration. (b) (c) Using the rule of supersymmetric configuration, we reach the brane setup which is good for the mirror transformation. (d) The brane setup of the mirror theory.

5.1 $\mathrm{Sp}'(1)$ with 3 fundamental flavors

In this example, the theory is Sp(1) gauge group with three hypermultiplets and two half-hypermultiplets. The moduli space has $d_v = 1$ and $d_H = 3 \times 2 + 2 \times 2/2 - 3 = 5$. The steps for finding the mirror theory are drawn in figure 12. First we go to the Higgs branch. Now equations (2.4) and (2.5) allow us the break the D3-branes between the 1/2NS-branes and the neighboring 1/2D5-brane as part (a). Form part (a) we move 1/2NS-brane inside to pass one 1/2D5-brane and get part (b). In this step, the 1/2NS-branes get rid of the D3-brane ending on them already. However, this brane setup does not readily give the correct mirror theory and we need go to the next step, i.e., moving 1/2NS-brane one step further inside as in part (c). Finally, we make the S-duality transformation and get the mirror theory in part (d). The mirror theory is $\mathrm{SO}(2) \times \mathrm{Sp}(1) \times \mathrm{SO}(3) \times \mathrm{Sp}(1) \times \mathrm{SO}(2)$ with four bifundamentals and two half-hypermultiplets one for each Sp(1) gauge group. We can check the moduli spaces of the mirror theory as having $d_v = 5$ and $d_H = (2 \times 4 + 2 \times 6 + 2 \times 2)/2 - (2 + 3 \times 3) = 12 - 11 = 1$.

5.2 The general case

Now we discuss the general case, i.e., $\mathrm{Sp}'(k)$ with n hypermultiplets and two half-hypermultiplets. The moduli spaces have $d_v = k$ and $d_H = 2nk + 2k - k(2k + 1) = 2nk - k(2k - 1)$. The main steps to get the mirror theory are in figure 13.

We can read out the mirror theory from the quiver diagram in part (b) and check the moduli space as having

$$\begin{aligned} d_v &= 4\sum_{t=1}^{k} t + k(2n - 4k - 1) = 2nk - k(2k - 1) \\ d_H &= \frac{1}{2}\left[2\sum_{t=1}^{k-1}((2t)^2 + 2t(2t + 2)) + 2(2k)^2 + (2n - 4k)2k(2k + 1) + 2(2k)\right] - \\ &\quad - \left[2\sum_{t=1}^{k-1}(t(2t - 1) + t(2t + 1)) + 2k(2k - 1) + (2n - 4k + 1)k(2k + 1)\right] \\ &= k \end{aligned} \tag{5.1}$$

As the case of Sp gauge group when $n = 2k$, the mirror theory has only one $\mathrm{Sp}(k)$ gauge group and the two half-hypermultiplets combine together to give one flavor for $\mathrm{Sp}(k)$. It means that we have a “hidden FI-term” in the original theory. However, it is not the end of the story. By careful observation, we find that when $n = 2k - 1$, the mirror theory has only one $\mathrm{SO}(2k)$ gauge group and two half-hypermultiplets also combine together to give one flavor for $\mathrm{SO}(2k)$ (this happens because in this case, we do not need move 1/2NS-brane one further step as we did from part (b) to part (c) in figure 12). So we get a “hidden FI-term” in this case also. This is not expected initially because it seems that for $n = 2k - 1$ we can not get the complete

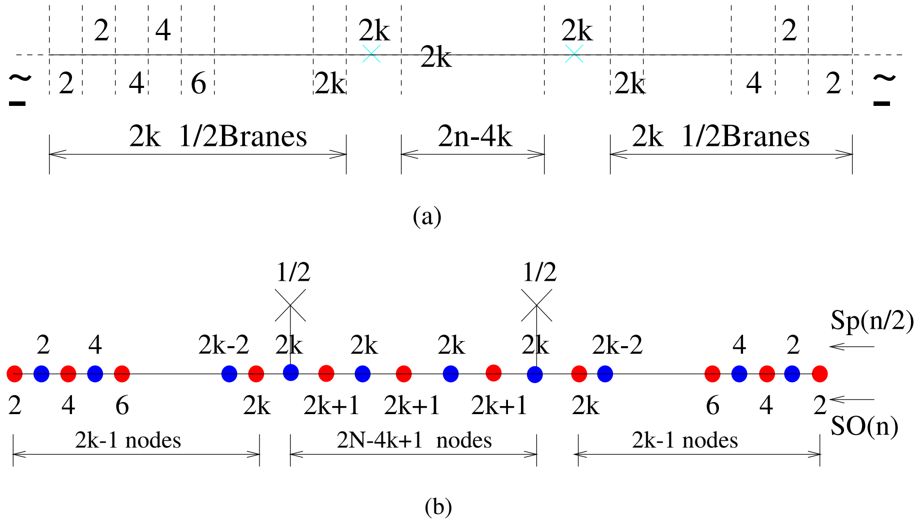


Figure 13: (a) The Higgs branch of $\mathrm{Sp}'(k)$ with N flavors after moving the two 1/2NS-branes inside. As before, the numbers above and below mean the number of 1/2D3-branes which connect two neighboring 1/2D5-branes. We can also consider it as the brane setup of the mirror theory just by considering the vertical line as 1/2NS-brane instead of 1/2D5-brane and the crosses as 1/2D5-branes instead of 1/2NS-branes. (b) The quiver diagram of the mirror theory. The numbers written above the (blue) node denote the Sp gauge group and the numbers written under the (red) node denote the SO gauge group.

Higgs branch, but this is not true. By studying the part (a) of figure 12, we find that for $n = 1$ in $\mathrm{Sp}'(1)$ we indeed get complete higgsing. Furthermore by the discussion in the next subsection we will see more clearly the reason why $n = 2k - 1$ gives a "hidden FI-term".

Now let us discuss the global symmetry. The results are very similar to those at the end of section 4. In the general case we have global $\mathrm{SO}(2N+1)$ flavor symmetry.[5] When $N = 2k-1$, the global symmetry goes to $\mathrm{SO}(2N+1) \times \mathrm{Sp}(1)$. When $N = 2k$, the global symmetry goes to $\mathrm{SO}(2N + 1) \times \mathrm{SO}(2)$.

5.3 Comparing the mirror of $\mathrm{Sp}(k)$ and $\mathrm{Sp}'(k)$

In the above, we have discussed the mirror of two kinds of Sp gauge groups, i.e., $\mathrm{Sp}(k)$ and $\mathrm{Sp}'(k)$. We want to ask ourselves whether there is any relation between the mirrors of these two Sp gauge groups? By checking the two quivers in figure 11 and figure 13, we find that these two quivers are exactly the same, except that N flavors in $\mathrm{Sp}'(k)$ should correspond to $N + 1$ flavors in $\mathrm{Sp}(k)$. This is reasonable

[5]From the discussion in the next subsection, the mirrors of single $\mathrm{Sp}'(k)$ with N flavors and single $\mathrm{Sp}(k)$ with $N+1$ flavors are identical. In the latter case, the flavor symmetry is $\mathrm{SO}(2N+2)$, but in the former case, we see only an obvious $\mathrm{SO}(2N + 1)$ flavor symmetry. However, in current situation of product gauge theories the argument of section 5.3 can not be applied directly. There is true distinguishing between $\mathrm{Sp}(k)$ and $\mathrm{Sp}'(k)$ gauge theories.

JHEP11(2000)033

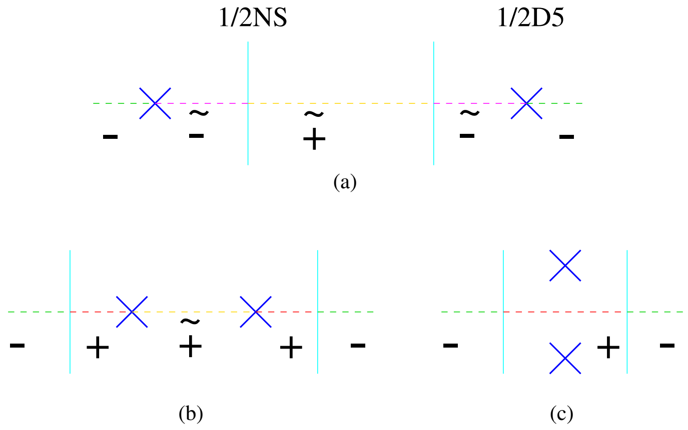


Figure 14: (a) We move one 1/2D5-brane from the left and right infinity. (b) By using the s-configuration in section 2, we change the position of 1/2D5-branes inside. (c) By combining the two 1/2D5-brane we get one physical D5-brane which can be moved off the $O3^+$-plane. From it we see that we change $\mathrm{Sp}'(k)$ to $\mathrm{Sp}(k)$ with one additional flavor.

because for $\mathrm{Sp}'(k)$ with N flavors there are two half-hypermultiplets which give the same degrees of freedom as one flavor. However, in principle there is a difference between one flavor and two half-hypermultiplets: for the former we can involve one mass parameter, but for the latter there is no such mass parameter. We will show, in the case of $\mathrm{Sp}'(k)$, that the two half-hypermultiplets do combine to give one flavor with the mass parameter. To see this, we move one 1/2D5-brane from infinity at each side to pass the 1/2NS-brane. By using the s-configuration in section 2, we get the brane setup of $\mathrm{Sp}(k)$ with an additional flavor. The whole discussion is shown in figure 14. Furthermore, it is easily to show that the two cases where a "hidden FI-parameter" shows in $\mathrm{Sp}(k)$ and $\mathrm{Sp}'(k)$ exactly match each other.

6. The mirror of $\mathrm{SO}(2k)$ gauge theory

After the discussion of the mirror theories for $\mathrm{Sp}(k)$ gauge groups, we now discuss $\mathrm{SO}(2k)$. There are no known results for the mirror of $\mathrm{SO}(2k)$ gauge groups and it is the main motivation of this paper to calculate it using the $O3$ plane. As in the last two sections, we first present the simple case of $\mathrm{SO}(2)$ with three flavors, then give the general results for $\mathrm{SO}(2k)$ with N flavors.

6.1 $\mathrm{SO}(2)$ with 3 flavors

For $\mathrm{SO}(2)$ gauge theory with three flavors, the moduli spaces have $d_v = 1$ and $d_H = 3 \times 2 - 1 = 5$. The steps for the mirror transformation are given in figure 15. In part (a) , we break the D3-branes by preserving the supersymmetric configurations, then use (2.4) and (2.5) to move the 1/2NS-brane passing the 1/2D5-brane to

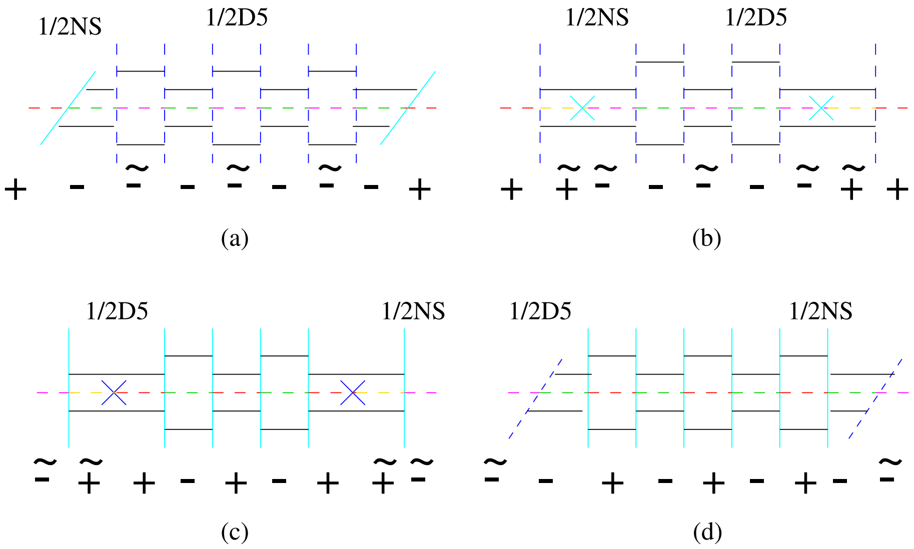


Figure 15: (a) The Higgs branch of SO(2) with three flavors. Notice how we split the D5-brane to satisfy the supersymmetric configurations. (b) Using the rule of supersymmetric configuration, we move the 1/2NS-brane one step to reach the brane setup which is convenient for the mirror transformation. (c) The brane setup of the mirror theory. (d) By moving the 1/2D5-brane one step outside, we combine the two half-hypermultiplets into one hypermultiplet.

get part (b). Unlike the Sp case, part (b) is already convenient for the mirror transformation, so we can make S-duality directly and get part (c) . From the brane setup in part (c) we read out the mirror theory to be $\mathrm{Sp}(1)\times \mathrm{SO}(2)\times \mathrm{Sp}(1)\times \mathrm{SO}(2)\times \mathrm{Sp}(1)$ with four bifundamentals and two half-hypermultiplets for the leftmost Sp(1) and two half-hypermultiplets for the rightmost Sp(1). Here we want to emphasize that in the two half-hypermultiplets for the leftmost Sp(1), one comes from the 1/2D5-brane and the other from the $\widetilde{O3}^-$ projection (same for the rightmost Sp(1)). That the half-hypermultiplets come from different sources is a general phenomenon in SO$(2k)$. However, for our simple example, we can combine these two half-hypermultiplets together by moving the 1/2D5-brane in part (c) to go part (d). Now we have one flavor of Sp(1) given by one physical D3-brane stuck between the 1/2NS-brane and the 1/2D5-brane. We need to emphasize that because the physical D3-brane is stuck between the 1/2NS-brane and the 1/2D5-brane, it does not contribute to the mass parameter. It will be interesting to compare it with the discussion in section 4.3, where we find that the two half-hypermultiplets of $\mathrm{Sp}'(k)$ can combine to give a flavor with free mass parameter. Finally, we calculate the dimension of the moduli spaces of mirror theory as $d_v = 5$ and $d_H = (4\times 4/2 + 2\times 2) - (2 + 3\times 3) = 12 - 11 = 1$.

6.2 An exotic example: SO(2) with 2 flavors

In this subsection, we discuss the mirror of SO(2) with two flavors. This theory will show one nontrivial phenomenon. The moduli are $d_v = 1$ and $d_H = 2\times 2 - 1 = 3$.

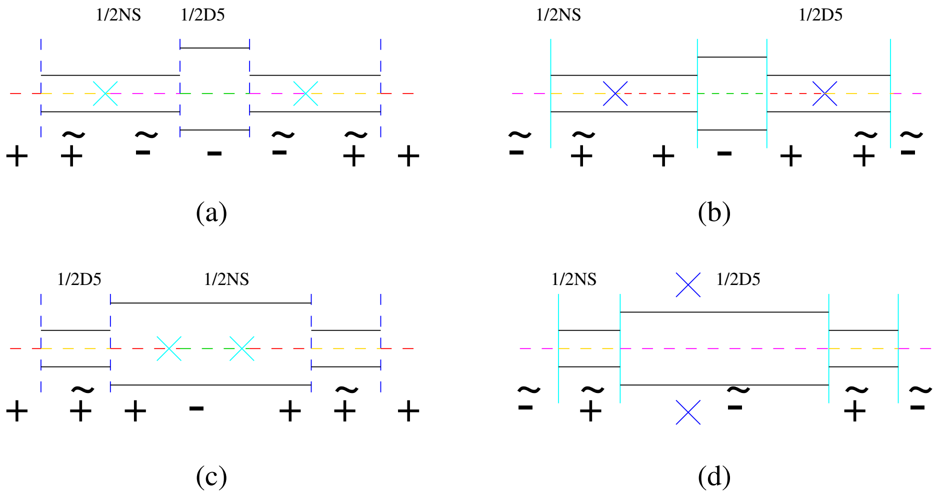


Figure 16: (a) The Higgs branch of SO(2) with two flavors. (b) By S-duality, we get the mirror theory as $\mathrm{Sp}(1) \times \mathrm{SO}(2) \times \mathrm{Sp}(1)$ with two bifundamentals and four half-hypermultiplets for the two Sp(1) gauge theories. (c) However, for this special case, it seems we can get another mirror theory by moving the 1/2NS-brane one further step inside from part (a) to part (c). In our case, now two 1/2NS-branes are in same interval. If they can combine together and leave the $O3^+$ plane, we can make the S-duality to get part (d). (d) The mirror theory got from part (c) is $\mathrm{Sp}(1) \times \mathrm{SO}(3) \times \mathrm{Sp}(1)$ with two bifundamentals , two half-hypermultiplets for two Sp(1) and one fundamental for SO(3).

According to the standard procedure introduced in the last subsection we get the Higgs branch as part (a) in figure 16 and the mirror theory in part (b). The dimensions of moduli spaces of the mirror theory in part (b) are $d_v = 1 + 1 + 1 = 3$ and $d_H = 2 \times 4/2 + 4 \times 2/2 - (3 + 3 + 1) = 9 - 8 = 1$.

However it seems we can get another possible Higgs branch in part (c) by moving the 1/2NS-brane one further step inside from part (a). If these two 1/2NS-branes do not meet together, the brane setup is not convenient to perform S-duality to get the mirror theory and we must go back to part (a). But in this special example, these two 1/2NS-branes do meet together. Now if these two 1/2NS-brane can combine to leave the $O3^+$ plane, we do get another mirror theory like part (d). Let us assume it is correct first and calculate the moduli spaces. In the part (d), the mirror theory is $\mathrm{Sp}(1) \times \mathrm{SO}(3) \times \mathrm{Sp}(1)$ with two bifundamentals, two half-hypermultiplets for the two Sp(1) and one fundamental for SO(3), so the moduli are $d_v = 3$ and $d_H = 2 \times 6/2 + 2 \times 2/2 + 3 - (3 + 3 + 3) = 11 - 9 = 2$. Therefore the results do not match. There is another inconsistent result because in the mirror theory of part (d) we get one "hidden FI-term" which does not exist in the mirror theory of part (b).

What is the resolution for the above inconsistency? Notice the combination of two 1/2NS-branes on the $O3^+$ is S-dual to the combination of two 1/2D5-branes on the $\widetilde{O3}^-$. We have discussed this configuration in section 3.1, where we showed, only when there is an extra physical D3-brane between these two 1/2D5-branes (1/2NS-

JHEP11(2000)033

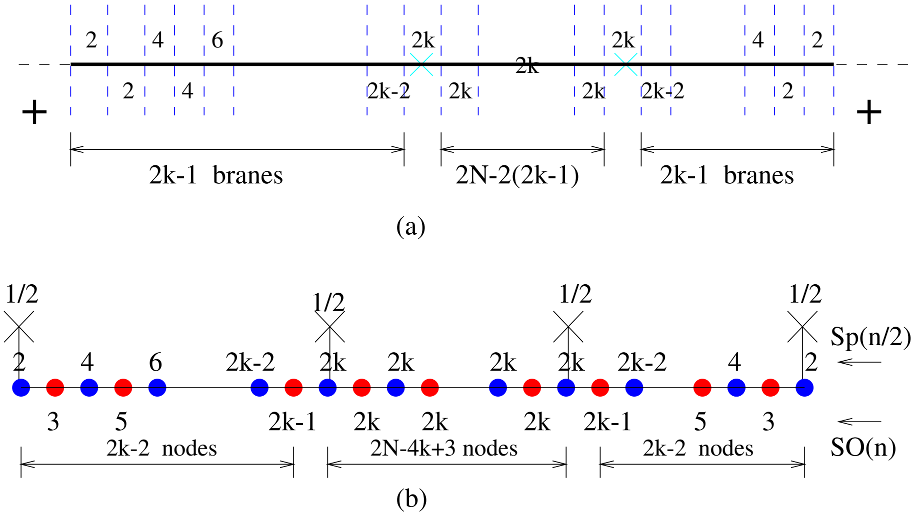


Figure 17: (*a*) The Higgs branch of SO($2k$) with N flavors in the setup of the D5-brane splitting. The numbers in the interval denote the number of 1/2D3-branes connecting the two neighboring 1/2D5-branes. (*b*) The quiver diagram of the mirror theory of SO($2k$) with N flavors. Notice that the index above the node means Sp($n/2$) and index below the node means SO(n). The 1/2 means the half-fundamental.

branes) can they combine and leave the O3-plane. So the conclusion is that the two 1/2NS-branes in part (c) can not combine and leave the O3-plane. We are left with only one correct mirror theory in part (b).

6.3 The general SO($2k$) with N flavors

With the experience of the SO(2) case, we can now work on the general SO($2k$) with N flavors. The moduli for this theory are $d_v = k$ and $d_H = 2kN - k(2k-1)$. The steps for the mirror theory are given in figure 17.

Again, we first break the D3-branes according to the supersymmetric configuration, then move the 1/2NS-branes inside to go to part (a). The brane setup in part (a) can be considered as the brane setup of S-duality just by exchanging the roles of the 1/2NS-brane and the 1/2D5-brane and putting in a proper O3-plane. For clarity, we draw the quiver diagram of the mirror theory in part (b). Let us check the result again by calculating the moduli of the mirror theory as

$$d_v = 4\sum_{n=1}^{k-1} n + k(2N-4k+3) = 2kN - k(2k-1) \tag{6.1}$$

$$d_H = \left[\frac{1}{2}\times 2\sum_{n=1}^{2k-2}(n+1)(n+2) + \frac{1}{2}4k^2(2N-4k+2) + \frac{1}{2}(2k+2k+2+2)\right] -$$
$$- \left[4\sum_{n=1}^{k-1} n(2n+1) + k(2k+1)(N-2k+2) + k(2k-1)(N-2k+1)\right] = k\,.$$

By checking part (a) in figure 17, we find that there is a "hidden FI-parameter" in the original theory when $N = 2k-1$ because two 1/2NS-branes will meet in same interval of $\widetilde{O3}^+$ plane. For general N, k, the global symmetry is an $\mathrm{Sp}(N)$ flavor symmetry, but in the case $N = 2k-1$ it is enhanced to $\mathrm{Sp}(N) \times \mathrm{SO}(3)$. We want to point out that there is only one case where *"hidden FI-parameters"* show in $\mathrm{SO}(2k)$ while for $\mathrm{Sp}(k)$ and $\mathrm{Sp}'(k)$ there are two cases. This difference can be seen very clearly in part (c) of figure 16. In that case two 1/2NS-branes do meet in same interval, but they can not combine and leave $O3^+$-plane. So there is no *"hidden FI-parameters"*.

7. The mirror of $\mathrm{SO}(2k+1)$ gauge theory

In this section, we discuss the mirror theories of $\mathrm{SO}(2k+1)$ to complete our study of single gauge groups. We first present the simple example of $\mathrm{SO}(3)$ with two flavors, then give the general results for $\mathrm{SO}(2k+1)$ with N flavors.

7.1 $\mathrm{SO}(3)$ with 2 flavors

For $\mathrm{SO}(3)$ with two flavors, the dimensions of moduli space are $d_v = 1$ and $d_H = 2 \times 3 - 3 = 3$. The steps to get the mirror theory are shown in figure 18. In part (a) we split the physical D5-branes into the 1/2D5-branes according the rules given in section two. In such a process we see the generation of two physical D3-branes

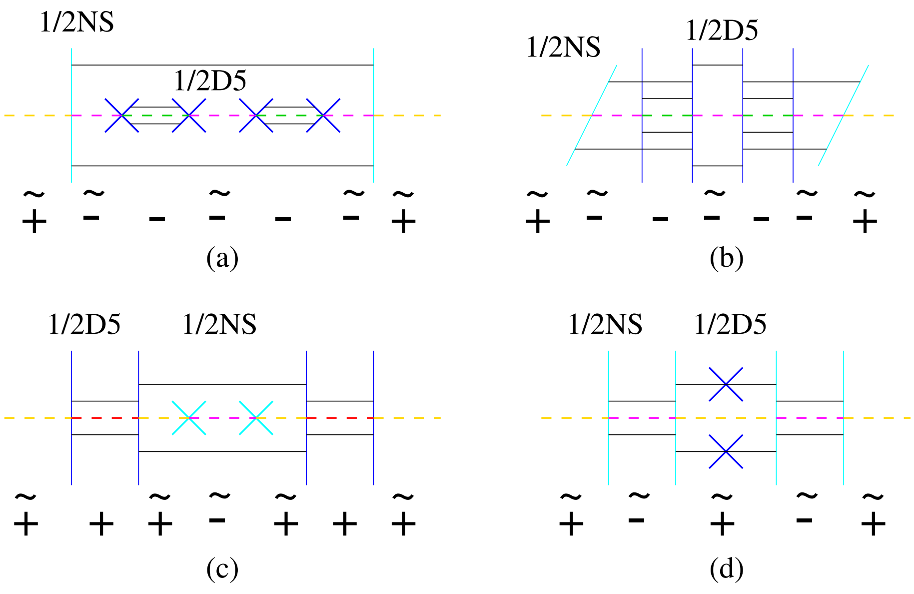


Figure 18: The mirror of $\mathrm{SO}(3)$ with two flavors. (a) Splitting of D5-branes according the rules given above. Notice the generation of D3-branes between 1/2D5-branes. (b) The Higgs branch of $\mathrm{SO}(3)$ theory. (c) By moving 1/2NS-branes inside we get rid of D3-brane ending on 1/2NS-branes and ready to go to the mirror theory. (d) The mirror theory. However, here we combine two 1/2D5-branes to give one physical D5-brane.

which is necessary to account for the correct Higgs branch. In part (b) we split the D3-brane between the 1/2D5-branes and 1/2NS-branes to go to the Higgs branch. Notice that there is no D3-branes connecting 1/2NS-brane and the nearest 1/2D5-brane which is required by s-rule. In part (c) we move the 1/2NS-branes inside to get rid of the D3-branes ending on them. Now we can make S-duality to give the mirror theory in part (d). However, in our example, there is a special property: two 1/2D5-branes can combine together and leave the $\widetilde{O3}^+$-plane to give one flavor.

Now we can read out the mirror theory as $\mathrm{SO}(3)\times\mathrm{Sp}(1)\times\mathrm{SO}(3)$ with two bifundamentals and one flavor for Sp(1). Let us calculate the dimension of moduli space. For the Coulomb branch, we have $d_v = 1+1+1 = 3$ which matches the Higgs branch of the original theory. For the Higgs branch, naively we should have $d_H = [\frac{1}{2}(6+6)+2]-[3+3+3] = -1$. However, the dimension can never be negative. The negative result means that our naive calculation is wrong. The reason is that in our naive calculation we assumed that there is complete higgsing. However, in our example, there is no complete higgsing in the mirror theory. After higgsing, we still keep two SO(2) gauge groups which give the correct $d_H = [8]-[9-2] = 1$ and match the Coulomb branch in the original theory. Furthermore, in our example, we have one flavor in the mirror theory which means that there is a "hidden FI-term" in the original theory.

7.2 The general case: $\mathrm{SO}(2k+1)$ with N flavors

Now let us discuss the mirror of $\mathrm{SO}(2k+1)$ with N flavors. The dimensions of moduli spaces are $d_v = k$ and $d_H = (2k+1)N - k(2k+1)$. The steps to get the mirror theory are given in figure 19. In part (a), we give the brane setup of the Higgs branch. In fact, we can consider it as well as the brane setup of the mirror theory by just changing the role of the vertical line and cross line (in Higgs branch, vertical lines denote 1/2D5-branes and cross lines, 1/2NS-branes; in the mirror theory, vertical lines denote 1/2NS-branes and cross lines, 1/2D5-branes). For convenience, we give the quiver diagram of the mirror theory in part (b).

Let us calculate the dimensions of the moduli spaces of the mirror theory to see if they match the dimensions of the moduli spaces of the original theory. The calculations are given as

$$\begin{aligned}
d_v &= \left[\sum_{i=1}^{k-1} 4i\right] + k(N-2k+3) + (k+1)(N-2k) \\
&= (2k+1)N - k(2k+1) \\
d_H &= 2\sum_{i=1}^{k-1}\left[\frac{(2i+1)2i}{2} + \frac{2i(2i+3)}{2} - 2i(2i+1)\right] + \\
&\quad + (2N-4k)\frac{2k(2k+2)}{2} - (N-2k+1)k(2k+1) - \\
&\quad - (N-2k)(k+1)(2(k+1)-1) + 2\frac{2k}{2} + 2\frac{2k(2k+1)}{2} - 2k(2k+1) + N
\end{aligned}$$

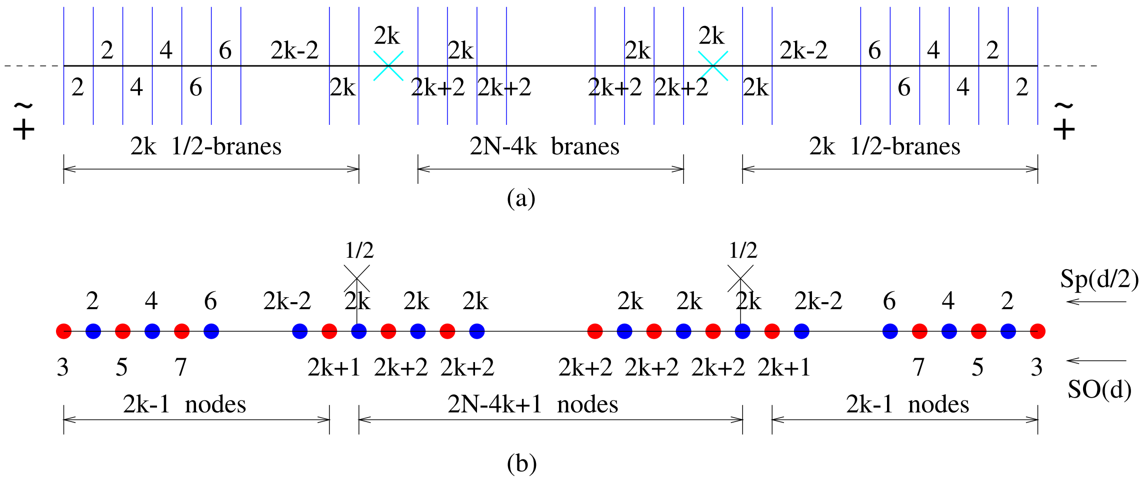


Figure 19: The mirror of SO$(2k+1)$ with N flavors. (a) The Higgs branch of the original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of the mirror theory.

$$\begin{aligned} &= [2k(k-1)] + [-(N-2k) - k(2k+1)] + [2k] + [N] \\ &= k \end{aligned} \tag{7.1}$$

Notice that we add N when we calculate d_H because after the higgsing, the mirror theory still keep N SO(2) gauge group. Furthermore, from the part (a) in figure 19 we see when $N = 2k$, two 1/2-branes can combine together and leave the orientifold plane. This means that when $N = 2k$ there is a *"hidden FI-term"* in the original theory. This also means that in the special case, the original theory has an enhanced global $\mathrm{Sp}'(N) \times \mathrm{SO}(3)$ symmetry instead of $\mathrm{Sp}'(N)$ flavor symmetry in general.

7.3 Comparing the mirrors of SO$(2k)$ and SO$(2k+1)$

At the end of this section, let us compare the mirror theories of SO$(2k)$ and SO$(2k+1)$. First we can start from the SO$(2k+1)$ with $N+1$ flavors to go to SO$(2k)$ with N flavors by higgsing one flavor. At the other side, by comparing the quivers in figure 17 and figure 19, it is obvious that if we change the SO(d) gauge group in figure 19 to SO$(d-2)$ while keeping the Sp$(d/2)$ gauge group we get exactly the quiver in figure 17. In particular, the two SO(3) gauge group in figure 19 go to SO(1) and disappear as a gauge group but add two half-hypermultiplets to two Sp(1) at the two ends of quiver in figure 17. This pattern can also be found if we higgs SO$(2k)$ with N flavors to SO$(2k-1)$ with $N-1$ flavors. In the latter case, we change the Sp$(d/2)$ gauge group in figure 17 to Sp$(d/2-1)$ gauge group while keeping SO(d) gauge group. After such a change, the quiver in figure 17 becomes exactly the quiver in figure 19 (the two nodes at the ends in figure 17 disappear). Notice that the higgsing in the original theory should correspond to the reduction of the Coulomb branch in the mirror theory. The change of gauge group is exactly the required reduction of the Coulomb branch in the mirror theory.

The above pattern passes another consistency check. Notice that for $\mathrm{SO}(2k+1)$ gauge theory with $N+1$ flavors, it has an enhanced $\mathrm{SO}(3)$ global symmetry when $N+1=2k$. After higgsing, we get $\mathrm{SO}(2k)$ with N flavors. For the latter, it has an enhanced $\mathrm{SO}(3)$ global symmetry exactly when $N=2k-1$. We see such hidden global symmetry is not broken by the Higgs mechanism as it should be.

8. The mirror of $\mathrm{Sp}(k)\times\mathrm{SO}(2m)$

We have discussed the mirror for a single Sp or SO group above. In this section, we generalize the above construction to the case of the product of Sp and SO gauge groups. Because after crossing the 1/2NS-brane $O3^{\pm}(\widetilde{O3^{\pm}})$ change to $O3^{\mp}(\widetilde{O3^{\mp}})$ and vise versa, we get two series of products $\mathrm{SO}(2n_1)\times\mathrm{Sp}(k_1)\times\mathrm{SO}(2n_2)\times\mathrm{Sp}(k_2)\ldots$ and $\mathrm{SO}(2n_1+1)\times\mathrm{Sp}'(k_1)\times\mathrm{SO}(2n_2+1)\times\mathrm{Sp}'(k_2)\ldots$. In this section, we discuss the first series and leave the second series to next section. For simplicity, we will discuss only the product of two gauge groups, i.e., $\mathrm{Sp}(k)\times\mathrm{SO}(2m)$ (the case of more product groups can be directly generalized). For this simple case, we still have two choices, the so called "elliptic model" [22] (X^6 direction is compactified) , or the "non-elliptic model" (X^6 direction is not compactified). We discuss these two models one by one.

8.1 The non-elliptic model

For the non-elliptic model, there are N fundamentals for $\mathrm{SO}(2m)$, H fundamentals for $\mathrm{Sp}(k)$ and one bifundamental (for simplicity we assume that N,H are sufficiently large. For N,H too small, there are a lot of special cases which need to be discussed individually and are tedious without providing too much new insight). The moduli are $d_v=m+k$ and $d_H=2mN+2kH+2mk-m(2m-1)-k(2k+1)$. In constructing the mirror theory, we need to study three cases: $m>k$, $m=k$ and $m<k$. Let us start with the case of $m>k$. The mirror theory is given in figure 20.

When we go to the Higgs branch, we can connect the D3-branes at the two sides of middle 1/2NS-brane. Because $m>k$, we can connect only k D3-branes such that they end on the left and the right 1/2NS-branes. There are still $m-k$ D3-branes ending on the middle 1/2NS-brane from the right. To get rid of those D3-branes, we must move the middle 1/2NS-brane to the right. The final Higgs branch after such a motion is given in part (a) of figure 20 and the quiver diagram of the mirror, in part (b). The moduli of the mirror can be calculated as

$$\begin{aligned} d_v = & \left[\sum_{p=1}^{p=k-1} 2p\right] + (2H-2k+1)k + \left[2\sum_{p=k+1}^{m-1} p\right] + \\ & + (2N-4m+2k+3)m + \left[2\sum_{p=1}^{p=m-1} p\right] \\ = & \; 2mN+2kH+2mk-m(2m-1)-k(2k+1) \end{aligned}$$

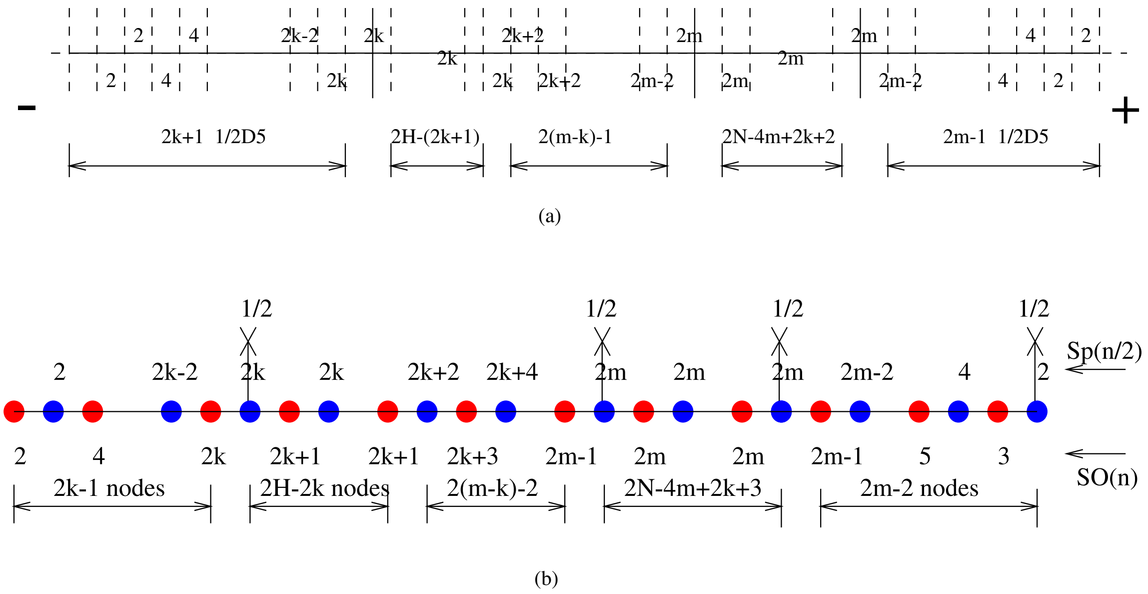


Figure 20: (a) The Higgs branch of $\mathrm{Sp}(k) \times \mathrm{SO}(2m)$ with N fundamentals for $\mathrm{SO}(2m)$, H fundamentals for $\mathrm{Sp}(k)$ and one bifundamental in the case of $m > k$. (b) The quiver diagram of the mirror theory of part (a). Notice that the index n above the node denotes $\mathrm{Sp}(n/2)$ and index n below the node denotes $\mathrm{SO}(n)$. The $1/2$ means half-hypermultiplets.

$$
\begin{aligned}
d_H = & \left[\sum_{i=1}^{i=k-1} \frac{1}{2}(2i \times 2i + 2i \times (2i+2)) - i(2i-1) - i(2i+1) \right] + \\
& + \left[\frac{4k^2}{2} + \frac{1}{2}(2H - 2k - 1)2k(2k+1) - (H-k)2k(2k+1) - k(2k-1) \right] + \\
& + \left[\sum_{i=1}^{m-k-1} \frac{1}{2}(2k+2i-1)(2k+2i) + \frac{1}{2}(2k+2i)(2k+2i+1) - \right. \\
& \quad \left. - 2(k+i)(2k+2i+1) \right] + \\
& + \left[\frac{2m(2m-1)}{2} + \frac{4m^2(2N - 4m + 2k + 2)}{2} - \right. \\
& \quad \left. - (N - 2m + k + 1)(m(2m-1) + m(2m+1)) - m(2m+1) \right] + \\
& + \left[\sum_{i=1}^{m-1} \frac{2i(2i+1)}{2} + \frac{(2i+1)(2i+2)}{2} - 2i(2i+1) \right] + \\
& + \left[\frac{2k}{2} + \frac{2m}{2} + \frac{2}{2} + \frac{2m}{2} \right] \\
= & [k(k-1)] + [-2k^2] + [(-k-m)(m-k-1)] + \\
& + [-2m] + [m^2 - 1] + [k + 2m + 1] \\
= & k + m
\end{aligned}
\tag{8.1}
$$

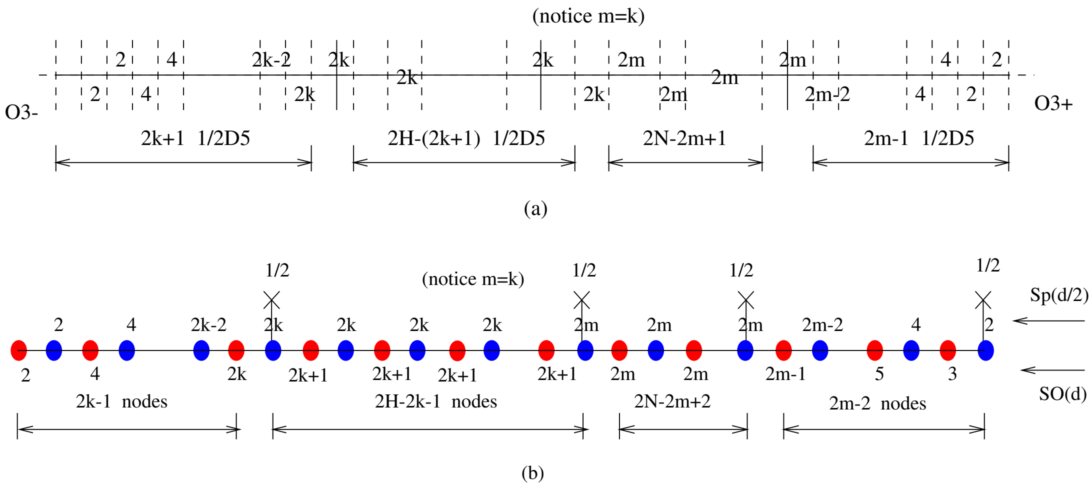


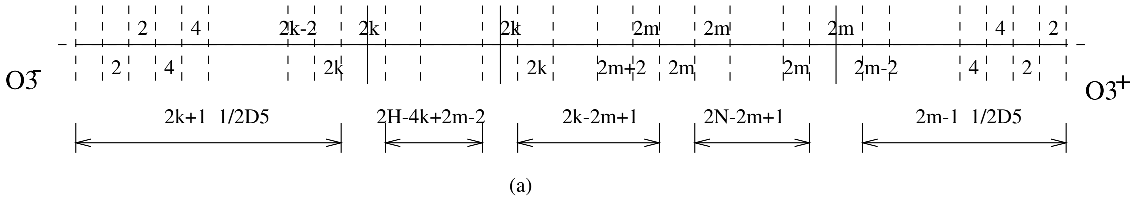
Figure 21: (a) The Higgs branch of Sp(k) × SO($2m$) with N fundamentals for SO($2m$), H fundamentals for Sp(k) and one bifundamental in the case of $m = k$. (b) The quiver diagram of the mirror theory of part (a). Notice that the index n above the node denotes Sp($n/2$) and index n below the node denotes SO(n). The 1/2 means the half-hypermultiplet.

From part (a) of figure 20, we see that when $2N - 4m + 2k + 2 = 0$, the two 1/2NS-branes meet together which indicates a "hidden FI-parameter" in the original theory.

After the discussion of the $m > k$ case, we go to the $m = k$ case. Here, by connecting the D3-branes between the two sides of the middle 1/2NS-brane, we get the Higgs branch looking like part (a) in figure 21.

From the quiver diagram part (b) we recalculate the moduli space as:

$$\begin{aligned}
d_v &= \left[\sum_{p=1}^{p=k-1} 2p\right] + (2H - 2k - 1 + 2N - 2m + 2 + 1)k + \left[2\sum_{p=k+1}^{m-1} p\right] \\
&= 2kN + 2kH - 2k^2 \\
&= 2mN + 2kH + 2mk - m(2m-1) - k(2k+1) \qquad \text{when } m = k\,, \\
d_H &= \left[\sum_{i=1}^{i=k-1} \frac{(2i \times 2i + 2i \times (2i+2))}{2} - i(2i-1) - i(2i+1)\right] + \\
&\quad + \left[\frac{4k^2}{2} + \frac{(2H-2k-2)2k(2k+1)}{2} - (2H-2k-1)k(2k+1) - k(2k-1)\right] + \\
&\quad + \left[(2N - 2m + 2)\frac{4m^2}{2} - (N - m + 1)(m(2m+1) + m(2m-1))\right] + \\
&\quad + \left[\sum_{i=1}^{m-1} \frac{2i(2i+1)}{2} + \frac{(2i+1)(2i+2)}{2} - 2i(2i+1)\right] + \\
&\quad + \left[\frac{2k}{2} + \frac{2m}{2} + \frac{2}{2} + \frac{2m}{2}\right] \\
&= [k(k-1)] + [-2k^2] + [0] + [m^2 - 1] + [k + 2m + 1] = k + m\,. \qquad (8.2)
\end{aligned}$$



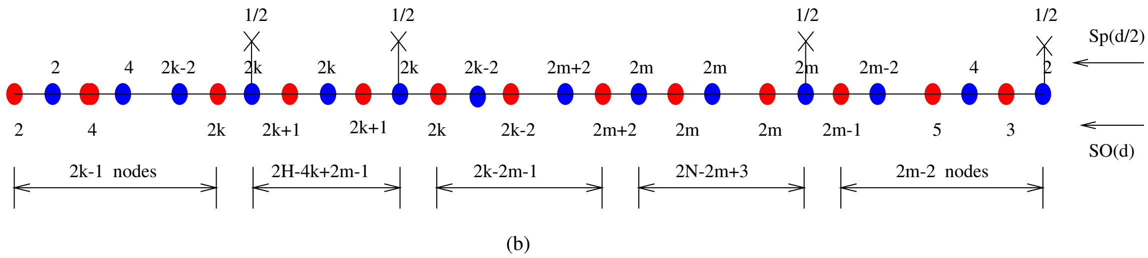


Figure 22: (a) The Higgs branch of $\mathrm{Sp}(k) \times \mathrm{SO}(2m)$ with N fundamentals for $\mathrm{SO}(2m)$, H fundamentals for $\mathrm{Sp}(k)$ and one bifundamental in the case of $m < k$. (b) The quiver diagram of the mirror theory of part (a). Notice that the index n above the node denotes $\mathrm{Sp}(n/2)$ and index n below the node denotes $\mathrm{SO}(n)$. The $1/2$ denotes the half-fundamental.

From the figure again, when $2H - 2k = 0$ or $2N - 2m + 2 = 0$, the two 1/2NS-branes meet together to give a "hidden FI-term" in the original theory.

Now we are left with only one case, i.e., $m < k$. In this last case, to get rid of the D3-branes, the middle 1/2NS brane should move to the left direction. The result is shown in figure 22.

The moduli of the mirror theory are

$$\begin{aligned}
d_v &= \left[\sum_{p=1}^{p=k-1} 2p\right] + (2H - 4k + 2m + 1)k + \left[2\sum_{p=m+1}^{k-1} p\right] + \\
&\quad +(2N - 2m + 3)m + \left[2\sum_{p=1}^{p=m-1} p\right] \\
&= 2mN + 2kH + 2mk - m(2m-1) - k(2k+1) \\
d_H &= \left[\sum_{i=1}^{i=k-1} \frac{(2i \times 2i + 2i \times (2i+2))}{2} - i(2i-1) - i(2i+1)\right] + \\
&\quad + \left[\frac{4k^2}{2} + (2H - 4k + 2m - 2)\frac{2k(2k+1)}{2} + \frac{4k^2}{2} - \right. \\
&\quad \left. - (2H - 4k + 2m - 1)k(2k+1) - 2k(2k-1)\right] + \\
&\quad + \left[\sum_{i=1}^{k-m-1} 2(m+i)\frac{2(m+i)}{2} + \frac{2(m+i)2(m+i+1)}{2} - \right. \\
&\quad \left. - (m+i)(2(m+i)-1) - (m+i)(2(m+i)+1)\right] +
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{2m(2m+2)}{2}+\frac{4m^2(2N-2m+2)}{2}-\right.\\
&\quad\left.-\ (N-m+1)(m(2m-1)+m(2m+1))-m(2m+1)\right]\\
&+\left[\sum_{i=1}^{m-1}\frac{2i(2i+1)}{2}+\frac{(2i+1)(2i+2)}{2}-2i(2i+1)\right]+\\
&+\left[\frac{2k}{2}+\frac{2k}{2}+\frac{2}{2}+\frac{2m}{2}\right]\\
=\ &[k(k-1)]+[-2k^2+k]+[(k+m)(k-m-1)]+\\
&+[m]+[m^2-1]+[2k+m+1]\\
=\ &k+m
\end{aligned}
\tag{8.3}
$$

There is also a possible "hidden FI-term" in original theory when $2H-4k+2m-2=0$ which can be explicitly seen in part (a) in figure 22.

8.2 The elliptic model

In the elliptic model, the X^6 direction is compactified such that for consistency, we must have an even number of 1/2NS-branes and an even number of gauge groups where half of them are Sp gauge groups and the other half, SO gauge groups. We discuss the case of only two gauge groups, i.e., $\mathrm{Sp}(k)\times\mathrm{SO}(2m)$ with H fundamentals for $\mathrm{Sp}(k)$, N fundamentals for $\mathrm{SO}(2m)$ and two bifundamentals. The moduli for this theory are $d_v=k+m$ and $d_H=2mN+2kH+4mk-m(2m-1)-k(2k+1)$. The mirror theory for the elliptic model is similar to the non-elliptic model. The only difference is that in the non-elliptic model we can connect the D3-branes only at the middle 1/2NS-brane, but here in the elliptic model we can connect the D3-branes to all 1/2NS-branes (here two 1/2NS-branes). Again we divide into three cases to discuss. The simple one is the case $m=k$. In this case, we can connect all D3-branes such that no D3-brane is left to end on the 1/2NS-branes. The Higgs branch and the quiver of the mirror are given in parts (a) (b) of figure 23 and the moduli are:

$$
\begin{aligned}
d_v &= (2H+2N)k\\
&= 2mN+2kH+4mk-m(2m-1)-k(2k+1) \qquad \text{when } m=k\\
d_H &= \left[\frac{(2H-2)2k(2k+1)}{2}-(H-1)2k(2k+1)-k(2k+1)\right]+\\
&\quad+\left[\frac{(2N+2)4k^2}{2}-N(k(2k+1)+k(2k-1))-k(2k-1)\right]+\left[\frac{2k}{2}+\frac{2k}{2}\right]\\
&= [-k(2k+1)]+[2k^2+k]+[2k]=k+m
\end{aligned}
\tag{8.4}
$$

There is still one case where the "hidden FI-term" appears, namely when $H=1$. In this case, two 1/2NS-branes in part (a) of figure 23 meet together.

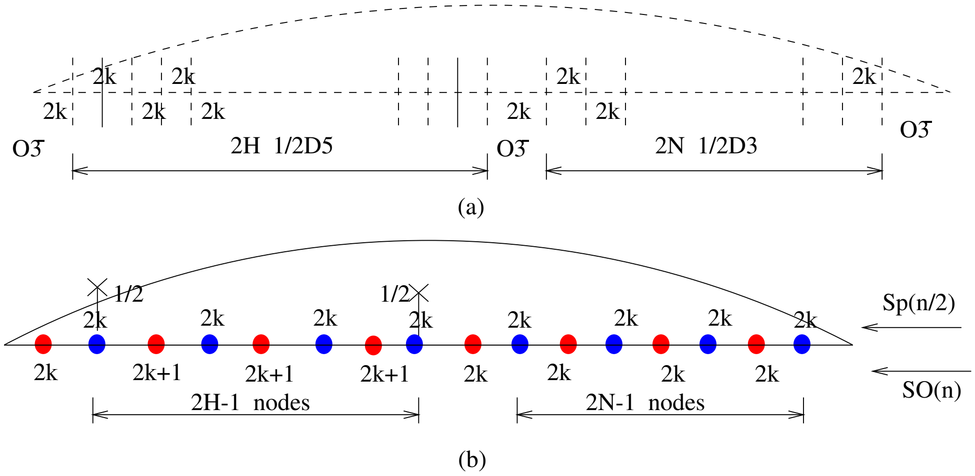


Figure 23: (*a*) The Higgs branch of elliptic Sp(k) $\times$ SO($2m$) with N fundamentals for SO($2m$), H fundamentals for Sp(k) and two bifundamentals in the case of $m = k$. The number here denotes how many 1/2D3-branes are connected to neighboring 1/2D5-branes. (*b*) The quiver diagram of the mirror theory of part (*a*). Notice that the index n above the node denotes Sp($n/2$) and index n below the node denotes SO(n). The 1/2 denotes the half hypermultiplets.

Now we move to the case of $k > m$. The Higgs branch looks like the superposition of the Higgs branch of the $m = k$ case together with that of a single Sp($k - m$) gauge theory. We give the result in figure 24. To check the result, we calculate the moduli as

$$\begin{aligned}
d_v &= \left[4 \sum_{i=1}^{i=k-m-1} (m+i)\right] + k(2H - 4(k-m) + 1) + m(2N+3) \\
&= 2mN + 2kH + 4mk - m(2m-1) - k(2k+1) \\
d_H &= \left[2 \sum_{i=1}^{i=k-m-1} \frac{2(m+i)2(m+i)}{2} + \frac{2(m+i)2(m+i+1)}{2} - \right. \\
&\quad \left. - (m+i)(2(m+i)-1) - (m+i)(2(m+i)+1)\right] + \\
&\quad + \left[2 \times \frac{4k^2}{2} + \frac{(2H - 4(k-m) - 2)2k(2k+1)}{2} - \right. \\
&\quad \left. - (2H - 4(k-m) - 1)k(2k+1) - 2k(2k-1)\right] + \\
&\quad + \left[\frac{(2N+2)4m^2}{2} + 2 \times \frac{2m(2m+2)}{2} - \right. \\
&\quad \left. - (N+1)(m(2m+1) + m(2m-1)) - m(2m+1)\right] + \left[2 \times \frac{2k}{2}\right] \\
&= [2(k^2 - m^2 - k - m)] + [-2k^2 + k] + [2m^2 + 3m] + [2k] = k + m\,. \qquad (8.5)
\end{aligned}$$

When $2H - 4(k-m) - 2 = 0$, two 1/2NS-branes will meet together in part (*a*) of figure 24. This is the condition that a "hidden FI-term" exists.

JHEP11(2000)033

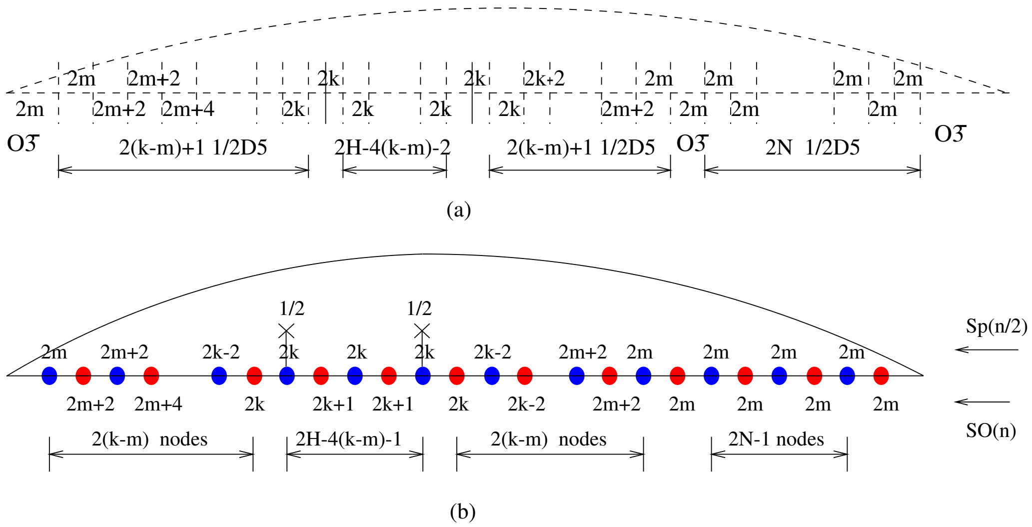


Figure 24: (a) The Higgs branch of elliptic $\mathrm{Sp}(k) \times \mathrm{SO}(2m)$ with N fundamentals for $\mathrm{SO}(2m)$, H fundamentals for $\mathrm{Sp}(k)$ and two bifundamentals in the case of $m < k$. The number here denotes how many 1/2D3-branes are connected to neighboring 1/2D5-branes. (b). The quiver diagram of the mirror theory of part (a). Notice that the index n above the node denotes $\mathrm{Sp}(n/2)$ and index n below the node denotes $\mathrm{SO}(n)$. The $1/2$ denotes the half hypermultiplets.

Now the remainder case is $m > k$. In this case, the Higgs branch looks like the superposition of two Higgs branches: that of the $m = k$ case and that of a single $\mathrm{SO}(2(m-k))$ theory. The result can be found in figure 25. The moduli of the mirror theory are

$$
\begin{aligned}
d_v &= \left[4 \sum_{i=1}^{i=m-k-1} (k+i)\right] + k(2H+1) + m(2N - 4(m-k) + 3) \\
&= 2mN + 2kH + 4mk - m(2m-1) - k(2k+1) \\
d_H &= \left[2 \sum_{i=1}^{i=m-k-1} \frac{2(k+i)(2k+2i+1)}{2} + \frac{(2k+2i+1)(2k+2i+2)}{2} - \right. \\
&\quad \left. - 2(k+i)(2k+2i+1)\right] \\
&\quad + \left[\frac{(2N-4(m-k)+2)4m^2}{2} - \right. \\
&\quad\quad \left. - (N - 2(m-k)+1)(m(2m+1) + m(2m-1)) - m(2m+1)\right] + \\
&\quad + \left[\frac{2h2k(2k+1)}{2} + \frac{2(2k+1)(2k+2)}{2} - (2H+1)k(2k+1)\right] + \left[2 \times \frac{2m}{2}\right] \\
&= [2m^2 - 2(k+1)^2] + [-2m^2 - m] + [2k^2 + 5k + 2] + [2k] \\
&= k + m
\end{aligned}
\tag{8.6}
$$

When $2N - 4(m-k) + 2 = 0$, there is a "hidden FI-term" in the original theory.

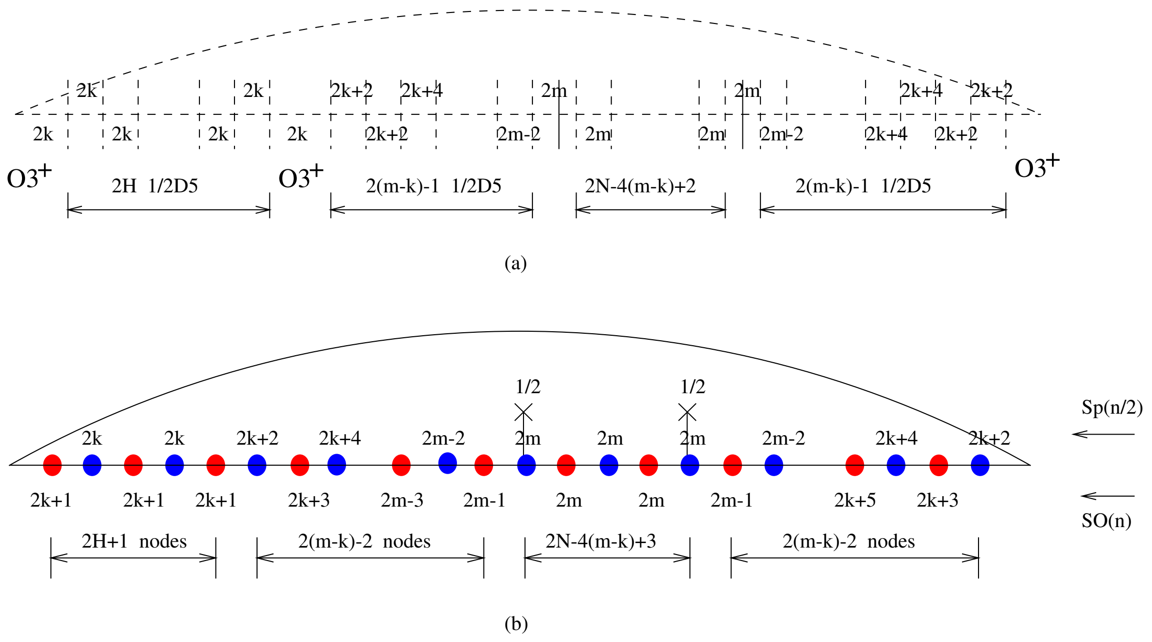


Figure 25: (a).The Higgs branch of elliptic $\mathrm{Sp}(k) \times \mathrm{SO}(2m)$ with N fundamentals for $\mathrm{SO}(2m)$, H fundamentals for $\mathrm{Sp}(k)$ and two bifundamentals in the case of $m > k$. The number here means how many 1/2D3-branes are connected to neighboring 1/2D5-branes. (b). The quiver diagram of the mirror theory of part (a). Notice that the index n above the node denotes $\mathrm{Sp}(n/2)$ and index n below the node denotes $\mathrm{SO}(n)$. The 1/2 denotes the half hypermultiplets.

9. The mirror of $\mathrm{Sp}'(k) \times \mathrm{SO}(2m+1)$

For completion, we give one more example: the mirror theory of $\mathrm{Sp}'(k) \times \mathrm{SO}(2m+1)$. We assume that there are H flavors for $\mathrm{Sp}'(k)$ gauge theory and N flavors for $\mathrm{SO}(2m+1)$ gauge theory. Besides, there are one or two bifundamentals and half-hypermultiplet for $\mathrm{Sp}'(k)$ depend on different situations. Again we divide our discussion into two parts: non-elliptic model and elliptic model.

9.1 The non-elliptic model

Let us start from the non-elliptic model. In this case, the dimensions of moduli spaces are $d_v = k + m$ and $d_H = 2kH + (2m+1)N + k(2m+1) + k - k(2k+1) - m(2m+1) = 2kH + (2m+1)N - 2k^2 + k - 2m^2 - m + 2km$ (here again, for simplicity we assume N, H are sufficiently large to avoid special cases). The mirror theory depends on whether $m > k$, $m = k$ or $m < k$. We first give the mirror of the case $m = k$ because in this particular case, we can combine the D3-branes at the two sides of middle 1/2NS-brane such that there is no D3-branes ending on the middle 1/2NS-brane anymore. The mirror theory is given in figure 26.

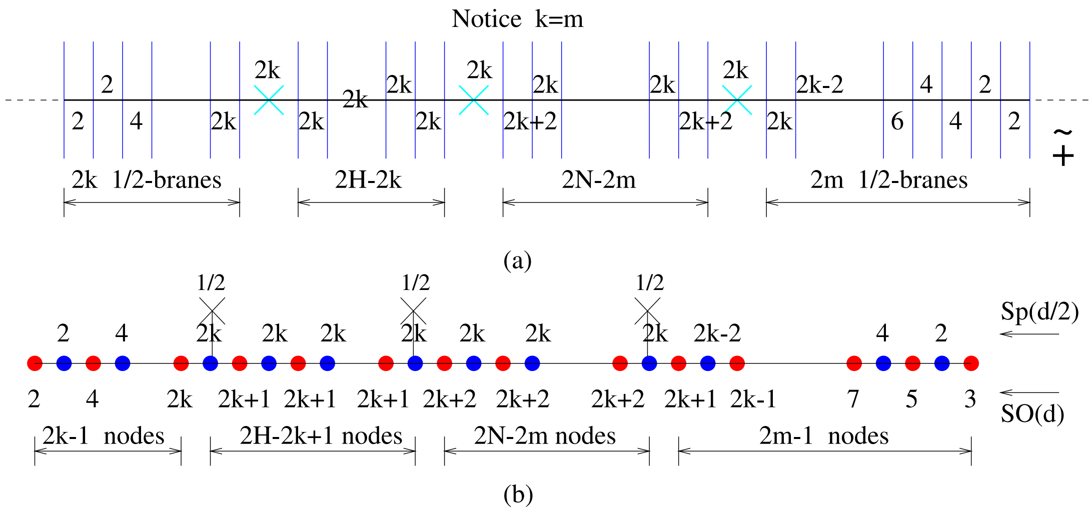


Figure 26: The mirror of $\mathrm{Sp}'(k) \times \mathrm{SO}(2m+1)$ with H flavors for $\mathrm{Sp}'(k)$, N flavors for $\mathrm{SO}(2m+1)$, a half-hypermultiplet for $\mathrm{Sp}'(k)$ and one bifundamental in case of $m = k$. (a) The Higgs branch of original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of mirror theory.

Let us check it by calculating the dimensions of moduli spaces of the mirror theory:

$$
\begin{aligned}
d_v &= 2\sum_{i=1}^{k-1} i + (2H-2k+2)k + (N-k)(k+k+1) + k + 2\sum_{i=1}^{k-1} i \\
&= 2kH + (2k+1)N - 2k^2 \\
d_H &= \sum_{i=1}^{k-1}\left[\frac{2i2i}{2} + \frac{2i(2i+2)}{2} - i(2i-1) - i(2i+1)\right] + \\
&\quad + \sum_{i=1}^{k-1}\left[\frac{(2i+1)2i}{2} + \frac{2i(2i+3)}{2} - 2i(2i+1)\right] + \\
&\quad + \frac{2k(2k+1)}{2}(2H-2k) - (H-k)2k(2k+1) + \\
&\quad + \frac{2k(2k+2)}{2}(2N_2k) - (N-k)(k(2k+1) + (k+1)(2k+1)) + \\
&\quad + \frac{2k2k}{2} - k(2k+1) - k(2k-1) + \frac{2k(2k+1)}{2} - k(2k+1) + 3\frac{2k}{2} + N \\
&= [k^2-k] + [k^2-k] + [0] + [-(N-k)] + [-2k^2] + [3k+N] \\
&= 2k,
\end{aligned} \tag{9.1}
$$

where when we calculate the d_H we add the term N to account for the remaining H SO(2) gauge groups after higgsing (this happens for latter examples so we will not mention it every time). When $2H - 2m = 0$ there is a *"hidden FI-term"* in the original theory.

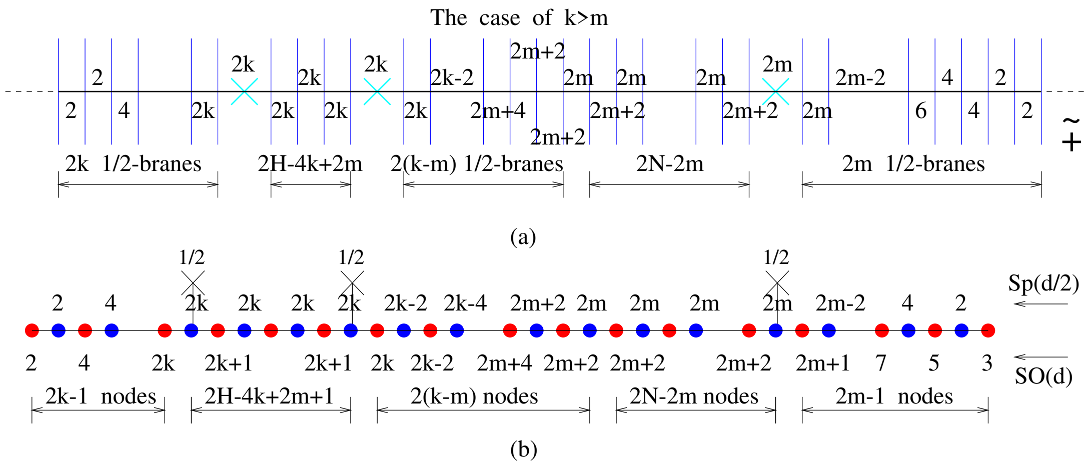


Figure 27: The mirror of $\mathrm{Sp}'(k) \times \mathrm{SO}(2m+1)$ with H flavors for $\mathrm{Sp}'(k)$, N flavors for $\mathrm{SO}(2m+1)$, a half-hypermultiplet for $\mathrm{Sp}'(k)$ and one bifundamental in case of $k > m$. (a) The Higgs branch of original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of mirror theory

Now we go to the case that $k > m$. In this case, after connecting the D3-branes at the two sides of the middle 1/2NS-brane, we still have $k - m$ D3-brane ending on the middle 1/2NS-brane from the left. The mirror theory is given in figure 27.

The dimensions of the moduli space of the mirror theory are

$$\begin{aligned}
d_v &= \left[2\sum_{i=1}^{k-1} i\right] + \left[2\sum_{i=1}^{m-1} i\right] + (2H - 4k + 2m + 3)k + \\
&\quad + \left[2\sum_{i=1}^{k-m-1}(m+i)\right] + [(N-m+2)m + (N-m)(m+1)] \\
&= [k^2 - k] + [m^2 - m] + [2kH - 4k^2 + 2km + 3k] + \\
&\quad + [k^2 - m^2 - k - m] + [(2m+1)N - 2m^2 + m] \\
&= 2kH + (2m+1)N - 2k^2 - 2m^2 + k - m + 2km \\
d_H &= \sum_{i=1}^{k-1}\left[\frac{2i2i}{2} + \frac{2i(2i+2)}{2} - i(2i-1) - i(2i+1)\right] + \\
&\quad + 2\frac{2k2k}{2} + \frac{2k(2k+1)}{2}(2H - 4k + 2m) - \\
&\quad - (2H - 4k + 2m + 1)k(2k+1) - 2k(2k-1) + \\
&\quad + \sum_{i=1}^{k-m-1}\left[\frac{(2(m+i))^2}{2} + \frac{2(m+i)2(m+i+1)}{2} - \right. \\
&\qquad \left. - (m+i)(2(m+i)-1) - (m+i)(2(m+i)+1)\right] + \\
&\quad + \frac{2m(2m+2)}{2}(2N - 2m + 1) - (N-m+1)m(2m+1) -
\end{aligned}$$

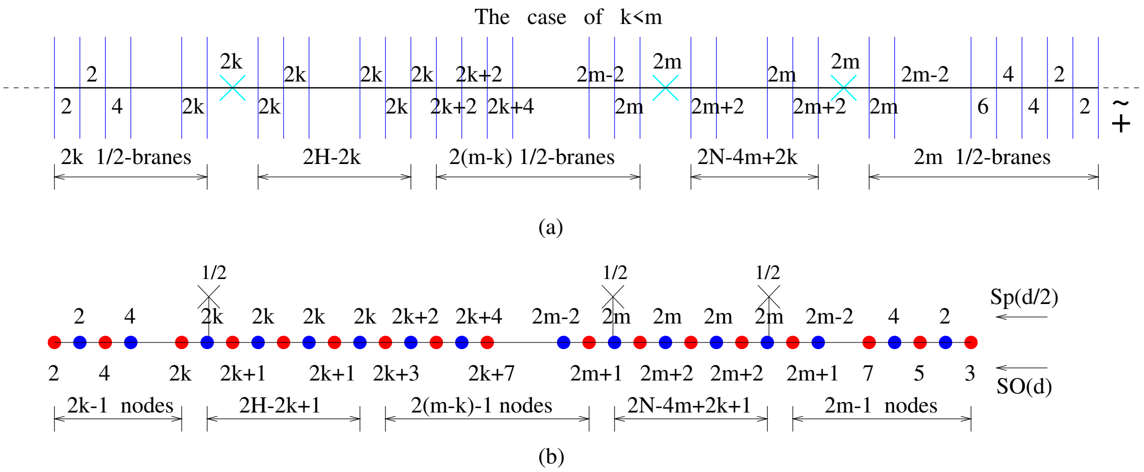


Figure 28: The mirror of $\mathrm{Sp}'(k) \times \mathrm{SO}(2m+1)$ with H flavors for $\mathrm{Sp}'(k)$, N flavors for $\mathrm{SO}(2m+1)$, a half-hypermultiplet for $\mathrm{Sp}'(k)$ and one bifundamental in case of $k < m$. (a) The Higgs branch of the original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of the mirror theory.

$$
\begin{aligned}
&- (N-m)m(2(m+1)-1) + \sum_{i=1}^{m-1} \left[\frac{(2i+1)2i}{2} + \frac{2i(2i+3)}{2} - 2i(2i+1) \right] + \\
&+ \frac{2m(2m+1)}{2} - m(2m+1) + 2\frac{2k}{2} + \frac{2m}{2} + N \\
&= [k^2 - k] + [-2k^2 + k] + [k^2 - m^2 - k - m] + \\
&+ [-N + 2m] + [m^2 - m] + [N + 2k + m] = k + m\,. \qquad (9.2)
\end{aligned}
$$

After the discussion of above two cases, we go to the last case: $k < m$. In this case, because $k < m$, after the combination of D3-branes at the two sides of middle 1/2NS-brane, we still leave $m-k$ D3-brane ending on it from the right. The mirror theory is given in figure 28.

Let us calculate the dimensions of moduli spaces:

$$
\begin{aligned}
d_v &= \left[2\sum_{i=1}^{k-1} i \right] + \left[2\sum_{i=1}^{m-1} i \right] + (2H - 2k + 2)k + \\
&\quad + 2\sum_{i=1}^{m-k-1} (k+i) + m(N - 2m + k + 3) + (m+1)(N - 2m + k) \\
&= [k^2 - k] + [m^2 - m] + [2kH - 2k^2 + 2k] + [m^2 - k^2 - k - m] + \\
&\quad + [(2m+1)N - 4m^2 + m + k + 2km] \\
&= 2kH + (2m+1)N + 2km - 2k^2 - 2m^2 - m + k \\
d_H &= \sum_{i=1}^{k-1} \left[\frac{2i2i}{2} + \frac{2i(2i+2)}{2} - i(2i-1) - i(2i+1) \right] +
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-1}\left[\frac{(2i+1)2i}{2}+\frac{2i(2i+3)}{2}-2i(2i+1)\right]+\\
&+\frac{(2k)^2}{2}+\frac{2k(2k+1)}{2}(2H-2k)-k(2k-1)-(2H-2k+1)k(2k+1)+\\
&+\sum_{i=1}^{m-k-1}\left[\frac{(2k+2i+1)2(k+i)}{2}+\frac{2(k+i)(2k+2i+3)}{2}-\right.\\
&\qquad\left.-\ 2(k+i)(2(k+i)+1)\right]+\\
&+\left[2\frac{2m(2m+1)}{2}+\frac{2m(2m+2)}{2}(2N-4m+2k)-\right.\\
&\quad\left.-\ (N-2m+k+3)m(2m+1)-(N-2m+k)(m+1)(2m+1)\right]+\\
&+\frac{2k(2k+3)}{2}+k+2m+N\\
=&\ [k^2-k]+[m^2-m]+[-2k^2]+[m^2-k^2-k-m]+\\
&+[-N-2m^2+m-k]+[2k^2+4k+2m+N]=m+k\,. \qquad (9.3)
\end{aligned}
$$

9.2 The elliptic model

In this section, we discuss the mirror theory of $\mathrm{Sp}'(k)\times\mathrm{SO}(2m+1)$ in the elliptic model. Now because X^6 is compact, the matter contents are H flavors for $\mathrm{Sp}'(k)$, N flavors for $\mathrm{SO}(2m+1)$ and two bifundamentals. The dimensions of the moduli spaces are $d_v=k+m$ and $d_H=2kH+(2m+1)N+2k(2m+1)-k(2k+1)-m(2m+1)=2kH+(2m+1)N+4km-2k^2-2m^2-m+k$. Again, our investigation will be divided into three cases $k=m$, $k>m$ and $k<m$.

Let us start from the case $k=m$. In this case, because we can combine all D3-branes at the two sides of 1/2NS-branes, it makes the mirror theory very simple as shown in figure 29.

Let us check the dimensions of moduli spaces:

$$
\begin{aligned}
d_v &= 2kH+Nk+N(k+1)=2kH+(2k+1)N\\
d_H &= \left[\frac{2k(2k+1)}{2}2H-2Hk(2k+1)\right]+[N]+\left[2\frac{2k}{2}\right]+\\
&\quad+\left[\frac{2k(2k+2)}{2}2N-Nk(2k+1)-N(k+1)(2k+1)\right]\\
&= 2k\,. \qquad (9.4)
\end{aligned}
$$

Now we go to the case of $k>m$. In this case, After combining the D3-branes, we still leave $k-m$ D3-branes in the interval of $\widetilde{O3}^{+}$-plane. The mirror theory is given in figure 30.

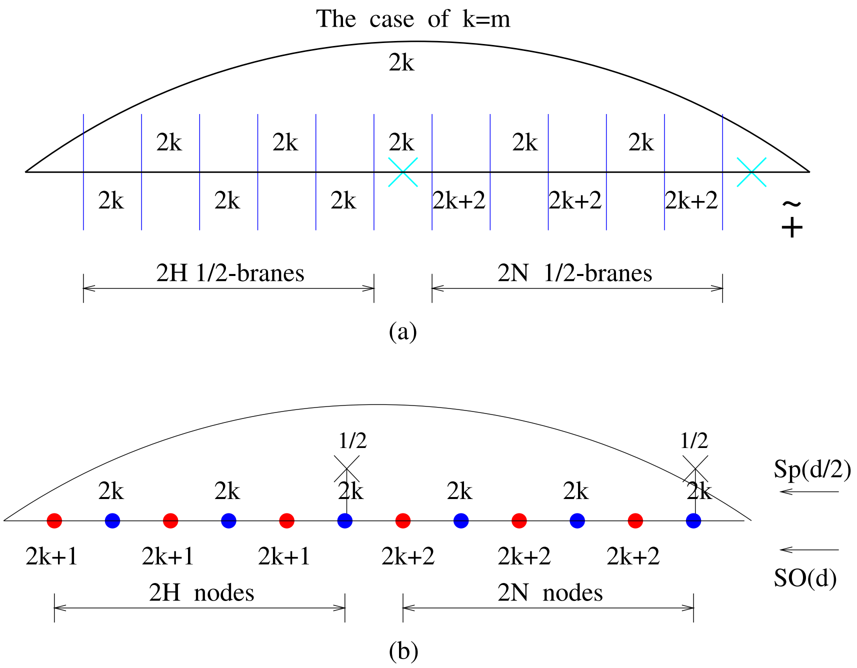


Figure 29: The mirror of $\mathrm{Sp}'(k) \times \mathrm{SO}(2m+1)$ with H flavors for $\mathrm{Sp}'(k)$, N flavors for $\mathrm{SO}(2m+1)$ and two bifundamentals in case of $k = m$. (a) The Higgs branch of the original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of the mirror theory.

The dimensions of moduli spaces are

$$
\begin{aligned}
d_v &= 2\sum_{i=1}^{k-m-1} 2(m+i) + k(2H-4k+4m+3) + m(N+1) + (m+1)N \\
&= [2k^2 - 2m^2 - 2k - 2m] + [2kH + (2m+1)N + 4km - 4k^2 + 3k + m] \\
&= 2kH + (2m+1)N + 4km - 2k^2 - 2m^2 + k - m \\
d_H &= 2\sum_{i=1}^{k-m-1}\left[\frac{(2(m+i))^2}{2} + \frac{2(m+i)2(m+i+1)}{2} - \right. \\
&\qquad\qquad \left. -(m+i)(2m+2i-1) - (m+i)(2m+2i+1)\right] + \\
&\quad + 2\frac{(2k)^2}{2} + \frac{2k(2k+1)}{2}(2H-4k+4m) - \\
&\quad - (2H-4k+4m+1)k(2k+1) - 2k(2k-1) + 2k + N \\
&= [2k^2 - 2m^2 - 2k - 2m] + [-2k^2 + k] + [-N + 2m^2 + 3m] + [N + 2k] \\
&= k + m
\end{aligned}
\tag{9.5}
$$

We are left only one more example, i.e., the case of $k < m$. For this case, after the combination, we still have $m - k$ D3-branes in the interval of $\widetilde{O3}^-$-plane. The mirror theory is given in figure 31.

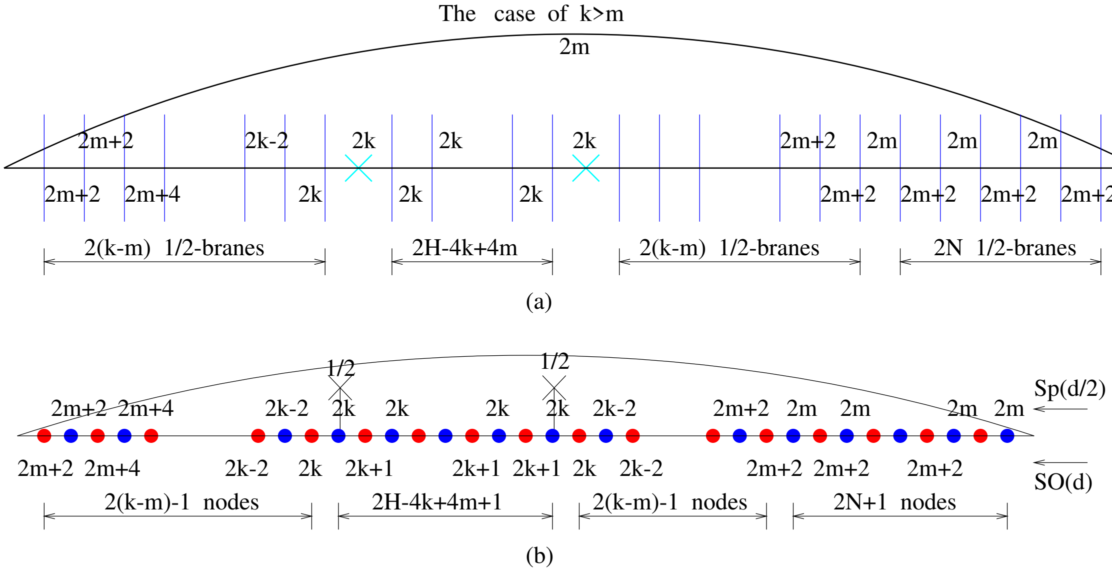


Figure 30: The mirror of $\mathrm{Sp}'(k) \times \mathrm{SO}(2m+1)$ with H flavors for $\mathrm{Sp}'(k)$, N flavors for $\mathrm{SO}(2m+1)$ and two bifundamentals in case of $k > m$. (a) The Higgs branch of the original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of the mirror theory.

The dimensions of moduli spaces are

$$
\begin{aligned}
d_v &= 2\sum_{i=1}^{m-k-1} 2(k+i) + k(2H+1) + \\
&\quad + m(N-2m+2k+3) + (m+1)(N-2m+2k) \\
&= [2m^2 - 2k^2 - 2k - 2m] + [2kH + (2m+1)N + 4km + m + 3k] \\
&= 2kH + (2m+1)N - 2m^2 - 2k^2 - m + k \\
d_h &= 2\sum_{i=1}^{m-k-1}\Bigg[\frac{(2k+2i)(2k+2i+1)}{2} + \frac{(2k+2i)(2k+2i+3)}{2} - \\
&\qquad - 2(k+i)(2k+2i+1)\Bigg] + \\
&\quad + [2\frac{2m(2m+1)}{2} + \frac{2m(2m+2)}{2}(2n-4m+4k) - \\
&\quad - (N-2m+2k+3)m(2m+1) - (N-2m+2k)(m+1)(2m+1)] + \\
&\quad + \frac{2k(2k+1)}{2}(2H) - (2H+1)k(2k+1) + 2\frac{2k(2k+3)}{2} + \\
&\quad + 2m + N \\
&= [2m^2 - 2k^2 - 2k - 2m] + [-N - 2m^2 - 2k + m] + [2k^2 + 5k] + [2m + N] \\
&= k + m\,.
\end{aligned}
\tag{9.6}
$$

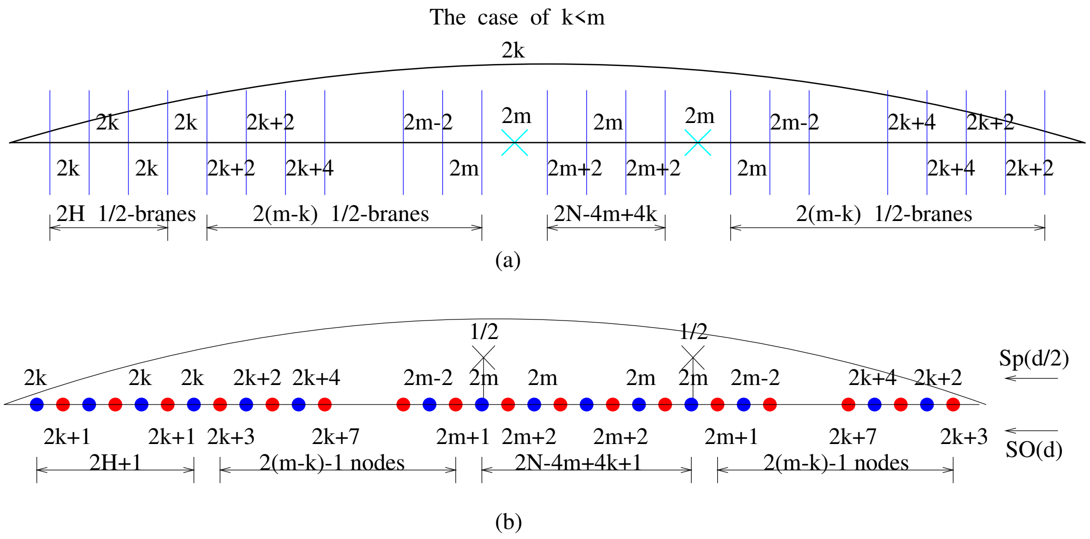


Figure 31: The mirror of $\mathrm{Sp}'(k) \times \mathrm{SO}(2m+1)$ with H flavors for $\mathrm{Sp}'(k)$, N flavors for $\mathrm{SO}(2m+1)$ and two bifundamentals in case of $k < m$. (a) The Higgs branch of the original theory or the Coulomb branch of the mirror theory. (b) The quiver diagram of the mirror theory.

10. Conclusion

In this paper, we give the mirror theories of $\mathrm{Sp}(k)$ and $\mathrm{SO}(n)$ gauge theories. In particular, for the first time the mirror of $\mathrm{SO}(n)$ gauge theory is given. In the construction of the mirror, we have made an assumption about the splitting of D5-branes on O3-planes in the *brane-plane* system.[6] We want to emphasize that because the splitting of D5-brane on O3-plane is a nontrivial dynamical process and we do not fully understand it at this moment, we can not really prove our assumption by calculation. However, although our discussions in this paper indicate that our assumption is consistent, the other independent checks are favorable. This gives one direction of further work as to prove our observation.

Furthermore, as we discussed in section three, our rules observed in this paper about the splitting of physical brane predict some nontrivial strong coupling limit of a particular field theory. It will be very interesting to use the Seiberg-Witten curve [23, 24] to show whether it is true.

There is another direction to pursue our investigation. By rotating one of the 1/2NS-branes [25, 26, 27, 2] we break the $N = 4$ theory in three dimensions to an $N = 2$ theory. Then we can discuss the mirror of $N = 2$ in three dimension as we have done in this paper. However, because there is less supersymmetry in the $N = 2$ case, things become more complex (for a detailed explanation of new features in $N = 2$, see [2]). Indeed, we can even break the supersymmetry further to discuss the mirror symmetry in the $N = 1$ case [28].

[6]For more details about the brane-plane system see [20].

Acknowledgments

We would like to extend our sincere gratitude to A. Kapustin, K. Intriligator and A. Uranga for fruitful discussions. Furthermore A. Hanany would like to thank A. Zaffaroni for discussion and B. Feng would like to thank Yang-Hui He, A. Naqvi, and J.S. Song for their suggestions and help.

This project is supported by the DOE under grant no. DE-FC02-94ER40818. A.H. is also partially supported by the National Science Foundation under grant no. PHY94-07194, by an A.P. Sloan Foundation Fellowship and by a DOE OJI award.

References

[1] K. Intriligator and N. Seiberg, *Mirror symmetry in three dimensional gauge theories*, *Phys. Lett.* **B 387** (1996) 513 [hep-th/9607207].

[2] J. de Boer, K. Hori, Y. Oz and Z. Yin, *Branes and mirror symmetry in $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys.* **B 502** (1997) 107 [hep-th/9702154].

[3] O. Aharony, A. Hanany, K. Intriligator, N. Seiberg and M.J. Strassler, *Aspects of $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys.* **B 499** (1997) 67 [hep-th/9703110].

[4] P.C. Argyres, M.R. Plesser and N. Seiberg, *The moduli space of $N = 2$ SUSY QCD and duality in $N = 1$ SUSY QCD*, *Nucl. Phys.* **B 471** (1996) 159 [hep-th/9603042].

[5] J. de Boer, K. Hori, H. Ooguri and Y. Oz, *Mirror symmetry in three-dimensional gauge theories, quivers and D-branes*, *Nucl. Phys.* **B 493** (1997) 101 [hep-th/9611063].

[6] J. de Boer, K. Hori, H. Ooguri, Y. Oz and Z. Yin, *Mirror symmetry in three-dimensional gauge theories,* $\mathrm{SL}(2,\mathbb{Z})$ *and D-brane moduli spaces*, *Nucl. Phys.* **B 493** (1997) 148 [hep-th/9612131].

[7] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[8] M. Porrati and A. Zaffaroni, *M-theory origin of mirror symmetry in three dimensional gauge theories*, *Nucl. Phys.* **B 490** (1997) 107 [hep-th/9611201].

[9] A. Kapustin, *D_n quivers from branes*, *J. High Energy Phys.* **12** (1998) 015 [hep-th/9806238].

[10] A. Hanany and A. Zaffaroni, *Issues on orientifolds: on the brane construction of gauge theories with* $\mathrm{SO}(2N)$ *global symmetry*, *J. High Energy Phys.* **07** (1999) 009 [hep-th/9903242].

[11] M. Aganagic and A. Karch, *Calabi-yau mirror symmetry as a gauge theory duality*, *Class. and Quant. Grav.* **17** (2000) 919 [hep-th/9910184].

JHEP11(2000)033

[12] A. Kapustin and M.J. Strassler, *On mirror symmetry in three dimensional abelian gauge theories*, *J. High Energy Phys.* **04** (1999) 021 [hep-th/9902033].

[13] N. Dorey and D. Tong, *Mirror symmetry and toric geometry in three dimensional gauge theories*, *J. High Energy Phys.* **05** (2000) 018 [hep-th/9911094].

[14] K. Hori, H. Ooguri and C. Vafa, *Non-abelian conifold transitions and $N = 4$ dualities in three dimensions*, *Nucl. Phys.* **B 504** (1997) 147 [hep-th/9705220].

[15] A. Giveon and D. Kutasov, *Brane dynamics and gauge theory*, *Rev. Mod. Phys.* **71** (1999) 983 [hep-th/9802067].

[16] A. Sen, *Duality and orbifolds*, *Nucl. Phys.* **B 474** (1996) 361 [hep-th/9604070].

[17] E. Witten, *Baryons and branes in anti de Sitter space*, *J. High Energy Phys.* **07** (1998) 006 [hep-th/9805112].

[18] J. Polchinski, *String theory*, vol. I chapter six, Cambridge University Press, 1998.

[19] A. Hanany, *Orientifold planes, points, and a supersymmetric index*, http://online.itp.ucsb.edu/online/susy99/hanany.

[20] A. Hanany and B. Kol, *On orientifolds, discrete torsion, branes and m theory*, *J. High Energy Phys.* **06** (2000) 013 [hep-th/0003025].

[21] N. Evans, C.V. Johnson and A.D. Shapere, *Orientifolds, branes and duality of 4d gauge theories*, *Nucl. Phys.* **B 505** (1997) 251 [hep-th/9703210].

[22] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166].

[23] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166].

[24] K. Landsteiner, E. Lopez and D.A. Lowe, *$N = 2$ supersymmetric gauge theories, branes and orientifolds*, *Nucl. Phys.* **B 507** (1997) 197 [hep-th/9705199]; *Supersymmetric gauge theories from branes and orientifold six-planes*, *J. High Energy Phys.* **07** (1998) 011 [hep-th/9805158].

[25] S. Elitzur, A. Giveon and D. Kutasov, *Branes and $N = 1$ duality in string theory*, *Phys. Lett.* **B 400** (1997) 269 [hep-th/9702014].

[26] S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici and A. Schwimmer, *Brane dynamics and $N = 1$ supersymmetric gauge theory*, *Nucl. Phys.* **B 505** (1997) 202 [hep-th/9704104].

[27] O. Aharony and A. Hanany, *Branes, superpotentials and superconformal fixed points*, *Nucl. Phys.* **B 504** (1997) 239 [hep-th/9704170].

[28] M. Gremm and E. Katz, *Mirror symmetry for $N = 1$ QED in three dimensions*, *J. High Energy Phys.* **02** (2000) 008 [hep-th/9906020].

JHEP11(2000)033